Optimal Actuator Location of the Minimum Norm Controls for Heat Equation with General Controlled Domain

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Abstract

In this paper, we study optimal actuator location of the minimum norm controls for a multi-dimensional heat equation with control defined in the space $L^p(0,T;L^2(\Omega))$. The actuator domain ω is quite general in the sense that it is required only to have a prescribed Lebesgue measure. A relaxation problem is formulated and is transformed into a two-person zero-sum game problem. By the game theory, we develop a necessary and sufficient condition and the existence of relaxed optimal actuator location for $p \in [2,+\infty]$, which is characterized by the Nash equilibrium of the associated game problem. An interesting case is for the case of $p=2$, for which it is shown that the classical optimal actuator location can be obtained from the relaxed optimal actuator location without additional condition. Finally for $p=2$, a sufficient and necessary condition for classical optimal actuator location is presented.

Keywords: Heat equation, optimal control, optimal location, game theory, Nash equilibrium.

AMS subject classifications: 35K05, 49J20, 65K210, 90C47, 93C20.

1 Introduction and main results

Different to lumped parameter systems, the location of actuator where optimal control optimizes the performance in systems governed by partial differential equations (PDEs) can often be chosen ([14]). Using a simple duct model, it is shown in [13] that the noise reduction performance depends strongly on actuator location. An approximation scheme is developed in [14] to find optimal location

of the optimal controls for abstract infinite-dimensional systems to minimize cost functional with the worst choice of initial condition. In fact, the actuator location problem has been attracted widely by many researchers in different contexts but most of them are for one-dimensional PDEs, as previously studied elsewhere [4, 6, 10, 11, 20, 22], to name just a few. Numerical research is one of the most important perspectives [4, 15, 18, 19, 23], among many others.

However, there are few results available in the literature for multi-dimensional PDEs. In [16], a problem of optimizing the shape and position of the damping set for internal stabilization of a linear wave equation in $\mathbb{R}^N, N = 1, 2$ is considered. The paper [17] considers a numerical approximation of null controls of the minimal L^∞-norm for a linear heat equation with a bounded potential. An interesting study is presented in [20] where the problem of determining a measurable subset of maximizing the L^2 norm of the restriction of the corresponding solution to a homogeneous wave equation on a bounded open connected subset over a finite time interval is addressed. In [9], the shape optimal design problems related to norm optimal and time optimal of null controlled heat equation have been considered. However, the controlled domains in [9] are limited to some special class of open subsets measured by the Hausdorff metric. The same limitations can also be found in shape optimization problems discussed in [7, 8]. Very recently, some optimal shape and location problems of sensors for parabolic equations with random initial data have been considered in [21].

In this paper, we consider optimal actuator location of the minimal norm controls for a multi-dimensional internal null controllable heat equation over an open bounded domain Ω in $\mathbb{R}^n$ space. Our internal actuator domains are quite general: They are varying over all possible measurable subsets ω of Ω where ω is only required to have a prescribed measure. This work is different from [21] yet one result (Theorem 1.3) can be considered as a refined multi-dimensional generalization of paper [19] where one-dimensional problem is considered.

Let us first state our problem. Suppose that $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a non-empty bounded domain with C^2-boundary $\partial\Omega$. Let $T > 0$, $y_0(\cdot) \in L^2(\Omega) \setminus \{0\}$, $a(\cdot, \cdot) \in L^\infty(\Omega \times (0, T))$, and $\alpha \in (0, 1)$. Denote by

$$\mathcal{W} = \left\{\omega \subset \Omega \,\middle|\, \omega \text{ is Lebesque measurable with } m(\omega) = \alpha \cdot m(\Omega)\right\}, \tag{1.1}$$

where $m(\cdot)$ is the Lebesgue measure on $\mathbb{R}^d$. For any $\omega \in \mathcal{W}$ and $p \in (1, +\infty]$, consider the following controlled heat equation

$$\begin{cases} y_t(x,t) - \Delta y(x,t) + a(x,t)y(x,t) = \chi_\omega(x)u(x,t) & \text{in } \Omega \times (0,T), \\ y(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $u(\cdot, \cdot) \in L^p(0, T; L^2(\Omega))$ is the control, and $\chi_\omega(\cdot)$ is the characteristic function of ω. For simplicity, we also denote $\chi_\omega(\cdot) \in \mathcal{W}$ when $\omega \in \mathcal{W}$. It is well known that for any $u(\cdot, \cdot) \in L^p(0, T; L^2(\Omega))$, Equation (1.2) admits a unique mild solution which is denoted by $y(\cdot; \omega, u)$.

The minimal norm control problem can be stated as follows. For a given time $T > 0$ and $\omega \in \mathcal{W}$, find a minimal norm control to solve the following optimal control problem:

$$(NP)_{p,\omega}: \; N_p(\omega) \triangleq \inf \left\{\|u\|_{L^p(0,T;L^2(\Omega))} \,\middle|\, y(T; \omega, u) = 0\right\}. \tag{1.3}$$

A classical optimal actuator location of the minimal norm control problem is to seek an $\bar{\omega} \in \mathcal{W}$ to minimize $N_p(\omega)$:

$$N_p(\bar{\omega}) = \inf_{\omega \in \mathcal{W}} N_p(\omega). \tag{1.4}$$

If such an $\bar{\omega}$ exists, we say that $\bar{\omega}$ is an optimal actuator location of the optimal minimal norm controls. Any $\bar{u} \in L^p(0,T;L^2(\Omega))$ that satisfies $y(T;\omega,\bar{u}) = 0$ and $\|\bar{u}\|_{L^p(0,T;L^2(\Omega))} = N_p(\bar{\omega})$ is called an optimal control with respect to the optimal actuator location $\bar{\omega}$.

The existence of optimal actuator location $\bar{\omega}$ is generally not guaranteed because of absence of the compactness of $\mathcal{W}$. For this reason, we consider instead a relaxed problem. Define

$$\mathcal{B} = \left\{ \beta \in L(\Omega;[0,1]) \;\Big|\; \int_\Omega \beta^2(x)\mathrm{d}x = \alpha \cdot \mathrm{m}(\Omega) \right\}, \tag{1.5}$$

where $L(\Omega;[0,1])$ consists of all Lebesgue measurable functions in Ω with values in $[0,1]$. Note that the set $\mathcal{B}$ is a relaxation to the set $\{\chi_\omega \mid \omega \in \mathcal{W}\}$ by observing that for any $\omega \in \mathcal{W}$, $\beta(\cdot) = \chi_\omega(\cdot) \in \mathcal{B}$, yet $\mathcal{B}$ is not anyhow the convex closure of $\{\chi_\omega \mid \omega \in \mathcal{W}\}$. Most often in what follows, we drop bracket by simply using β to denote the function $\beta(\cdot)$. This remark is also is applied to other functions in some places when there is no risk of arising the confusion.

For any $\beta \in \mathcal{B}$, consider the following system:

$$\begin{cases} y_t(x,t) - \Delta y(x,t) + a(x,t)y(x,t) = \beta(x)u(x,t) & \text{in } \Omega \times (0,T), \\ y(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1.6}$$

where once again the control $u(\cdot,\cdot) \in L^p(0,T;L^2(\Omega))$. Denote the solution of (1.6) by $y(\cdot;\beta,u)$ as counterpart of $y(\cdot;\omega,u)$ but with obvious different meaning. Accordingly, the problem $(NP)_{p,\omega}$ is changed into a relaxation problem of the following:

$$(NP)_{p,\beta}: \; N_p(\beta) \triangleq \inf \left\{ \|u\|_{L^p(0,T;L^2(\Omega))} \;\big|\; y(T;\beta,u) = 0 \right\}, \tag{1.7}$$

and the classical problem (1.4) is also relaxed to the following problem

$$N_p(\bar{\beta}) = \inf_{\beta \in \mathcal{B}} N_p(\beta). \tag{1.8}$$

Any solution $\bar{\beta}$ to problem (1.8) is called a relaxed optimal actuator location. If there is $\bar{\beta} = \chi_{\bar{\omega}}$ solves problem (1.8), then $\bar{\omega}$ is an optimal actuator location of the optimal minimal norm controls.

Our main approach is based on the two-person zero-sum game theory. If we are minimizing the cost with two variable functions $u(\cdot,\cdot)$ and $\beta(\cdot)$ one after another, then problem (1.8) can be written as

$$\inf_{\beta \in \mathcal{B}} \inf_{u \in D_\beta} \|u\|_{L^p(0,T;L^2(\Omega))} \text{ where } D_\beta = \{u \in L^p(0,T;L^2(\Omega)) \mid y(T;\beta,u) = 0\}. \tag{1.9}$$

This is a typical two-level optimization problem yet not a game problem. Indeed, in the framework of two-person zero-sum game theory, any Stackelberg game problem which is also called leader-follower game problem (see, e.g., [25]) should be of the form:

$$\inf_{x \in E} \sup_{y \in F} J(x,y) \text{ or } \sup_{y \in F} \inf_{x \in E} J(x,y),$$

where it is required that the set E is independent of the set F. It is interesting that we can use the relationship between problem $(NP)_{p,\beta}$ (1.7) and its dual problem which is a variational problem when $\beta = \chi_\omega$ ([12]) to transform the problem (1.9) into a Stackelberg game problem in the framework of two-person zero-sum game theory, which gives in turn the solution of our problem.

The main result of this paper is the following Theorem 1.1.

Theorem 1.1. *For any given $p \in [2, +\infty]$ with q being its conjugate exponent: $1/p + 1/q = 1$, there exists at least one solution to problem (1.8). In addition, $\bar{\beta}$ is a solution to problem (1.8) if and only if there is $\bar{\psi} \in \overline{Y_q}$ such that $(\bar{\beta}, \bar{\psi})$ is a Nash equilibrium of the following two-person zero-sum game problem: Find $(\bar{\beta}, \bar{\psi}) \in \mathcal{B} \times \overline{Y_q}$ such that*

$$\begin{aligned} \left[\frac{1}{2}\left\|\bar{\beta}\bar{\psi}(\cdot)\right\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \bar{\psi}(0)\,\rangle\right] &= \sup_{\beta \in \mathcal{B}} \left[\frac{1}{2}\left\|\beta\bar{\psi}(\cdot)\right\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \bar{\psi}(0)\,\rangle\right], \\ \left[\frac{1}{2}\left\|\bar{\beta}\bar{\psi}(\cdot)\right\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \bar{\psi}(0)\,\rangle\right] &= \inf_{\psi \in \overline{Y_q}} \left[\frac{1}{2}\left\|\bar{\beta}\psi(\cdot)\right\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \psi(0)\,\rangle\right]. \end{aligned} \tag{1.10}$$

where $\overline{Y}_q$ is defined in Definition 3.24 in section 3.2.3.

Remark 1.2. *The above necessary and sufficient condition is characterized by the Nash equilibrium of the associated game problem. Furthermore, for any solution $\bar{\beta}$ to problem (1.8), the set*

$$\left\{\hat{\psi} \in \overline{Y_q} \mid (\bar{\beta}, \hat{\psi}) \text{ is a Nash equilibrium}\right\} \tag{1.11}$$

is a singleton and independent of $\bar{\beta}$. Indeed, the set defined in (1.11) equals to $\{\bar{\psi}\}$ where $\bar{\psi}$ is the unique solution to problem (GP2) (3.44). Based on this fact, we can present a necessary condition to characterize any solution $\bar{\beta}$ in an alternative way in case the Nash equilibrium is not easy to be determined. That is, if $\bar{\beta}$ is a solution to problem (1.8), then $\bar{\beta}$ solves the following problem:

$$\sup_{\beta \in \mathcal{B}} \ \left\|\beta\bar{\psi}(\cdot)\right\|^2_{L^q(0,T;L^2(\Omega))}. \tag{1.12}$$

All results are illustrated in Remark 3.29 in section 3.2.3.

The case of $p = 2$ is of special interest. In this case, the solution of the classical problem (1.4) can be obtained from the associated relaxation problem (1.8).

Theorem 1.3. *Let $p = 2$ and let $\bar{\psi}$ be the unique solution to problem (GP2) (3.44). Then there exists at least one $\bar{\omega} \in \mathcal{W}$ such that*

$$N_2(\bar{\omega}) = \inf_{\omega \in \mathcal{W}} N_2(\omega) = N_2(\bar{\beta}) = \inf_{\beta \in \mathcal{B}} N_2(\beta),$$

where $\bar{\beta} = \chi_{\bar{\omega}}$. Moreover, $\bar{\omega}$ is an optimal actuator location of the optimal minimal norm controls if and only if $\bar{\omega}$ solves the problem following

$$\sup_{\omega \in \mathcal{W}} \left\|\chi_\omega \bar{f}\right\|^2_{L^2(\Omega)},$$

where $\bar{f}(x) = \displaystyle\int_0^T \bar{\psi}^2(x,t)\mathrm{d}t$.

We proceed as follows. In section 2, we formulate the problem (1.8) into a two-person zero-sum Stackelberg game problem. Several equivalent forms are presented. Section 3 is devoted to the proof of the main results, where in subsection 3.1, we discuss the relaxed problem, and in subsection 3.2 we discuss the associated two-person zero-sum game problem. Subsection 3.2.1 presents the existence of the relaxed optimal location, and subsection 3.2.2 discusses the value of the two-person zero-sum game. The Nash equilibrium is investigated in subsection 3.2.3. We end section 3.2 by presenting the proof of Theorem 1.1. In subsection 3.3, we discuss the case of $p = 2$. We conclude the context by presenting the proof of Theorem 1.3.

2 From relaxation problem to game problem

Suppose that the $p \in (1, +\infty]$ is fixed, $\beta \in \mathcal{B}$, and q is the conjugate exponent of p: $\frac{1}{p} + \frac{1}{q} = 1$. Now let us consider the dual problem of $(NP)_{p,\beta}$. Consider the dual system of (1.6):

$$\begin{cases} \varphi_t(x,t) + \Delta\varphi(x,t) - a(x,t)\varphi(x,t) = 0 & \text{in } \Omega \times (0,T), \\ \varphi(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(x,T) = z(x) & \text{in } \Omega, \\ y_\varphi(x,t) = \beta(x)\varphi(x,t) & \text{in } \Omega \times (0,T), \end{cases} \tag{2.1}$$

where $z \in L^2(\Omega)$ is given and $y_\varphi(x,t)$ is the output of (2.1). We denote the solution of (2.1) by $\varphi(\cdot; z)$.

Introduce the functional:

$$J(z; \beta, q) \triangleq \frac{1}{2} \|\beta\varphi(\cdot; z)\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \varphi(0; z)\,\rangle, \tag{2.2}$$

and propose the following variational problem:

$$(\text{Min } J)_{\beta,q} : \; V_q(\beta) = \inf_{z \in L^2(\Omega)} J(z; \beta, q). \tag{2.3}$$

The following Lemma 2.1 whose proof is presented at the end of section 3.1 gives a relation between problems $(NP)_{p,\beta}$ (1.7) and $(\text{Min } J)_{\beta,q}$ (2.3), which enable us to formulate the problem (1.8) into a two-person zero-game problem.

Lemma 2.1. *Let $\beta \in \mathcal{B}$ and $y_0 \in L^2(\Omega) \setminus \{0\}$. Let $N_p(\beta)$ and $V_q(\beta)$ be defined by (1.8) and (2.3) respectively. Then*

$$V_q(\beta) = -\frac{1}{2} N_p(\beta)^2. \tag{2.4}$$

Remark 2.2. *When $\beta = \chi_\omega$ for $\omega \in \mathcal{W}$, the corresponding equality (2.4) has been verified in [24]. Here we establish it for our relaxation problem.*

To transform problem (1.8) into a game problem by (2.4), we need to introduce two spaces. Let

$$Y = \{\varphi(\cdot; z) |\; z \in L^2(\Omega)\}, \tag{2.5}$$

where $\varphi(\cdot; z)$ is the solution of (2.1) with the initial value $z \in L^2(\Omega)$. Obviously, Y is a linear space from the linearity of PDE (2.1).

With Lemma 2.1 and space Y, it turns out immediately that the problem (1.8) is actually a minimax problem. Precisely, to solve problem (1.8), we only need to consider the following problem:

$$\sup_{\beta\in\mathcal{B}} -\frac{1}{2}N_p(\beta)^2 = \sup_{\beta\in\mathcal{B}} V_q(\beta). \tag{2.6}$$

By the definition of $V_q(\beta)$ in (2.3), the problem (2.6) is equivalent to the following problem:

$$\sup_{\beta\in\mathcal{B}} \inf_{z\in L^2(\Omega)} \left[\frac{1}{2}\left\|\beta\varphi(\cdot; z)\right\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \varphi(0,z)\,\rangle\right].$$

Furthermore, by the definition (2.5), the problem above is equivalent to the following problem:

$$\sup_{\beta\in\mathcal{B}} \inf_{\psi\in Y} \left[\frac{1}{2}\|\beta\psi\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \mathcal{T}_{\beta,q}(\beta\psi)\,\rangle\right],$$

where $\mathcal{T}_{\beta,q} : \beta\overline{Y}_{\beta,q} \to L^2(\Omega)$ is a compact operator which will be specified later in (3.11) with $Y \subset \overline{Y}_{\beta,q}$ and $\mathcal{T}_{\beta,q}(\beta\psi) = \psi(0)$ for any $\psi \in Y$. To sum up, we have obtained the following equivalences:

$$\inf_{\beta\in\mathcal{B}} N_p(\beta)$$

$$\Updownarrow$$

$$\sup_{\beta\in\mathcal{B}} -\frac{1}{2}N_p(\beta)^2 = \sup_{\beta\in\mathcal{B}} V_q(\beta)$$

$$\Updownarrow$$

$$\sup_{\beta\in\mathcal{B}} \inf_{z\in L^2(\Omega)} \left[\frac{1}{2}\left\|\beta\varphi(\cdot; z)\right\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \varphi(0,z)\,\rangle\right] \tag{2.7}$$

$$\Updownarrow$$

$$\sup_{\beta\in\mathcal{B}} \inf_{\psi\in Y} \left[\frac{1}{2}\|\beta\psi\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \mathcal{T}_{\beta,q}(\beta\psi)\,\rangle\right]. \tag{2.8}$$

Remark 2.3. *We note that if the optimal solutions to problems (1.8), (2.7), and (2.8) exist, then they are the same. In addition, the existence of solution to problem (2.7) means that there exists $\bar\beta \in \mathcal{B}$ such that*

$$\begin{aligned}
&\inf_{z\in L^2(\Omega)} \left[\frac{1}{2}\left\|\bar\beta\varphi(\cdot; z)\right\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \varphi(0,z)\,\rangle\right] \\
&= \sup_{\beta\in\mathcal{B}} \inf_{z\in L^2(\Omega)} \left[\frac{1}{2}\left\|\beta\varphi(\cdot; z)\right\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \varphi(0,z)\,\rangle\right].
\end{aligned}$$

Similar remark can be made to the solution of (2.8).

The problem (2.8) is a typical Stackelberg game problem which has the following equivalent form:

$$\text{(GP1):}\quad \sup_{\beta\in\mathcal{B}} \inf_{\psi\in Y} \left[\frac{1}{2}\|\beta\psi\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, \beta\mathcal{T}^*_{\beta,q} y_0, \psi\,\rangle\right]. \tag{2.9}$$

To solve the game problem arisen from problem (GP1) (2.9), we need to put into the framework of two-person zero-sum game theory. Let us recall some basic facts of the two-person zero-sum game problem. There are two players: Emil and Frances. Emil takes his strategy x from his strategy set E and Frances takes his strategy y from his strategy set F. Let $f : E \times F$ be the index cost function. Emil wants to minimize the function F while Frances wants to maximize F. In the framework of two-person zero-sum game, the solution to (2.9) is called a Stackelberg equilibrium. The most important concept for two-person zero-sum game is the Nash equilibrium.

Definition 2.4. *Suppose that E and F are strategy sets of Emil and Frances, respectively. Let $f : E \times F \mapsto \mathbb{R}$ be an index cost functional. We call $(\bar{x}, \bar{y}) \in E \times F$ to be a Nash equilibrium if,*

$$f(\bar{x}, y) \leq f(\bar{x}, \bar{y}) \leq f(x, \bar{y}), \quad \forall\, x \in E, y \in F.$$

The following result is well known, see, for instance, Proposition 8.1 of [1, p.121]. It connects the Stackelberg equilibrium with the Nash equilibrium.

Proposition 2.5. *The following conditions are equivalent.*

(i) $(\bar{x}, \bar{y})$ *is a Nash equilibrium;*

(ii) $V^+ = V^-$ *and $\bar{x}$ solves the following problem (or equivalently, $\bar{x}$ is a Stackelberg equilibrium associated):*

$$\inf_{x \in E} \sup_{y \in F} f(x, y), \text{ i.e. } \sup_{y \in F} f(\bar{x}, y) = V^+,$$

and $\bar{y}$ solves the following problem (or equivalently, $\bar{y}$ is a Stackelberg equilibrium associated):

$$\sup_{y \in F} \inf_{x \in E} f(x, y), \text{ i.e. } \inf_{x \in E} f(x, \bar{y}) = V^-,$$

where

$$V^+ \triangleq \inf_{x \in E} \sup_{y \in F} f(x, y), \quad V^- \triangleq \sup_{y \in F} \inf_{x \in E} f(x, y). \tag{2.10}$$

When $V^+ = V^-$, we say that the game problem attains its value $V^+ = V^-$.

Returning back to our problem (GP1) (2.9), it is seen that the index cost function is defined by

$$F(\theta, \psi) = -\frac{1}{2}\|\beta\psi\|^2_{L^q(0,T;L^2(\Omega))} - \langle \beta \mathcal{T}^*_{\beta,q} y_0,\, \psi \rangle, \tag{2.11}$$

The first player who controls the function $\beta \in \mathcal{B}$ wants to minimize F while the second player who controls the function $\psi \in Y$ wants to maximize F. Thus we can discuss problem (GP1) (2.9) in the framework of two-person zero-sum game.

3 Proof of the main results

First of all, let us recall the null controllability for the controlled system (1.6).

Lemma 3.1. *The system (1.6) is null controllable if and only if the dual system (2.1) is exactly observable: There exists positive constant $C_{q,\alpha}$ such that*

$$\|\varphi(0;z)\|_{L^2(\Omega)} \le C_{q,\alpha} \|\beta\varphi(\cdot;z)\|_{L^q(0,T;L^2(\Omega))}, \forall\, z \in L^2(\Omega) \text{ and } \beta \in \mathcal{B}. \tag{3.1}$$

The inequality of (3.1) is referred as the "observability inequality" for system (2.1).

Proof. When $\beta = \chi_\omega$, it is well known that system (1.6) is null controllable if and only if the "observability inequality" holds for dual system (2.1): There exists $\hat{C}_{q,b} > 0$ such that

$$\|\varphi(0;z)\|_{L^2(\Omega)} \le \hat{C}_{q,b} \|\chi_\omega\varphi(\cdot;z)\|_{L^q(0,T;L^2(\Omega))}, \forall\, z \in L^2(\Omega) \text{ and } m(\omega) \ge b > 0, \tag{3.2}$$

with some constant b. In addition, $\hat{C}_{q,b}$ is monotone decreasing with respect to b yet $\hat{C}_{q,b}$ is independent of ω. For any $\beta \in \mathcal{B}$, let

$$\lambda = \frac{m\left(\beta(x) \ge \sqrt{\alpha/2}\right)}{m(\Omega)}.$$

By

$$\begin{aligned}
&1 \cdot m\left(\beta(x) \ge \sqrt{\alpha/2}\right) + \alpha/2 \cdot m\left(\beta(x) < \sqrt{\alpha/2}\right) \\
\ge & \int_{\beta(x)\ge\sqrt{\alpha/2}} \beta^2(x)\mathrm{d}x + \int_{\beta(x)<\sqrt{\alpha/2}} \beta^2(x)\mathrm{d}x \\
= & \int_\Omega \beta^2(x)\mathrm{d}x = \alpha \cdot m(\Omega),
\end{aligned}$$

it follows that

$$\lambda \cdot m(\Omega) + \alpha/2(1-\lambda) \cdot m(\Omega) \ge \alpha \cdot m(\Omega).$$

Consequently, $\lambda \ge \dfrac{\alpha}{2-\alpha}$. We thus have

$$m\left(\beta(x) \ge \sqrt{\alpha/2}\right) \ge \frac{\alpha}{2-\alpha} \cdot m(\Omega),\ \forall\, \beta \in \mathcal{B}. \tag{3.3}$$

It then follows from (3.2) with $\omega = \{\beta(x) \ge \sqrt{\alpha/2}\}$ that

$$\begin{aligned}
\|\varphi(0;z)\|_{L^2(\Omega)} \le\ & \hat{C}_{q,\lambda} \left\|\chi_{\{\beta(x)\ge\sqrt{\alpha/2}\}}\varphi(\cdot;z)\right\|_{L^q(0,T;L^2(\Omega))} \\
\le\ & \hat{C}_{q,\lambda} \left\|\chi_{\{\beta(x)\ge\sqrt{\alpha/2}\}}\frac{\beta(x)}{\sqrt{\alpha/2}}\varphi(\cdot;z)\right\|_{L^q(0,T;L^2(\Omega))} \\
=\ & \frac{\sqrt{2}\hat{C}_{q,\lambda}}{\sqrt{\alpha}} \left\|\chi_{\{\beta(x)\ge\sqrt{\alpha/2}\}}\beta(x)\varphi(\cdot;z)\right\|_{L^q(0,T;L^2(\Omega))} \\
\le\ & \frac{\sqrt{2}\hat{C}_{q,\lambda}}{\sqrt{\alpha}} \|\beta(x)\varphi(\cdot;z)\|_{L^q(0,T;L^2(\Omega))} \\
=\ & \frac{\sqrt{2}\hat{C}_{q,\frac{\alpha}{2-\alpha}}}{\sqrt{\alpha}} \|\beta(x)\varphi(\cdot;z)\|_{L^q(0,T;L^2(\Omega))} \\
\triangleq\ & C_{q,\alpha} \|\beta\varphi(\cdot;z)\|_{L^q(0,T;L^2(\Omega))}.
\end{aligned}$$

This is (3.1). □

Remark 3.2. *Following from the proof of Lemma 3.1, the constant $C_{q,\alpha}$ in inequality (3.1) is independent of $\beta \in \mathcal{B}$.*

3.1 Relaxed case

To introduce the operator $\mathcal{T}_{\beta,q}$ in (2.8), we introduce two spaces first.

Lemma 3.3. *Let Y be defined by (2.5). For each $\beta \in \mathcal{B}$, define a function in Y by*

$$F_0(\varphi) = \|\beta\varphi\|_{L^q(0,T;L^2(\Omega))}, \ \forall\, \varphi \in Y.$$

Then (Y, F_0) is a linear normed space. We denote this normed space by $Y_{\beta,q}$.

Proof. It suffices to show that $F_0(\psi) = \|\beta\psi\|_{L^q(0,T;L^2(\Omega))} = 0$ implies $\psi = 0$. Actually, by (3.3),

$$\sqrt{\alpha/2}\|\chi_{\{\beta(x)\geq\sqrt{\alpha/2}\}}\psi\|_{L^q(0,T;L^2(\Omega))} \leq \|\beta\psi\|_{L^q(0,T;L^2(\Omega))} = 0.$$

By the unique continuation (see, e.g., [3]) for heat equation, we arrive at $\psi = 0$. □

Denote by

$$\overline{Y}_{\beta,q} = \text{ the completion of the space } Y_{\beta,q}. \tag{3.4}$$

It is usually hard to characterize $\overline{Y}_{\beta,q}$. However, we have the following description for $\overline{Y}_{\beta,q}$.

Lemma 3.4. *Let $1 \leq q < \infty$, $\beta \in \mathcal{B}$, and let $\overline{Y}_{\beta,q}$ be defined by (3.4). Then under an isometric isomorphism, any element of $\overline{Y}_{\beta,q}$ can be expressed as a function $\hat{\varphi} \in C([0,T); L^2(\Omega))$ which satisfies (in the sense of weak solution)*

$$\begin{cases} \hat{\varphi}_t(x,t) + \Delta\hat{\varphi}(x,t) - a(x,t)\hat{\varphi}(x,t) = 0 & \text{in } \Omega\times(0,T), \\ \hat{\varphi}(x,t) = 0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{3.5}$$

and $\beta\hat{\varphi} = \lim\limits_{n\to\infty}\beta\varphi(\cdot;z_n)$ for some sequence $\{z_n\} \subset L^2(\Omega)$ in $L^q(0,T;L^2(\Omega))$, where $\varphi(\cdot;z_n)$ is the solution of (2.1) with initial value $z = z_n$.

Proof. Let $\overline{\psi} \in (\overline{Y}_{\beta,q}, \bar{F}_0)$, where $(\overline{Y}_{\beta,q}, \bar{F}_0)$ is the completion of $(Y_{\beta,q}, F_0)$. By the definition, there is a sequence $\{z_n\}$ in $L^2(\Omega)$ such that

$$\bar{F}_0(\varphi(\cdot;z_n) - \overline{\psi}) \to 0,$$

from which, one has

$$F_0\left(\varphi(\cdot;z_n) - \varphi(\cdot;z_m)\right) = \bar{F}_0(\varphi(\cdot;z_n) - \varphi(\cdot;z_m)) \to 0 \text{ as } n, m \to \infty.$$

In other words,

$$\|\beta\varphi(\cdot;z_n) - \beta\varphi(\cdot;z_m)\|_{L^q(0,T;L^2(\Omega))} \to 0 \text{ as } n, m \to \infty. \tag{3.6}$$

Hence, there exists $\hat{\psi} \in L^q(0,T;L^2(\Omega))$ such that

$$\beta\varphi(\cdot;z_n) \to \hat{\psi} \ \text{ strongly in } \ L^q(0,T;L^2(\Omega)). \tag{3.7}$$

Let $\{T_k\} \subset (0,T)$ be such that $T_k \nearrow T$. i.e. T_k is strictly monotone increasing and converges to T. Denote $\varphi_n \equiv \varphi(\cdot; z_n)$.

(a). For T_1, by the observability inequality (3.1), and (3.6),

$$\begin{aligned} \|\varphi(T_2; z_n)\|_{L^2(\Omega)} &\leq C(1)\|\beta\varphi(\cdot; z_n)\|_{L^q(T_2,T;L^2(\Omega))} \\ &\leq C(1)\|\beta\varphi(\cdot; z_n)\|_{L^q(0,T;L^2(\Omega))} \leq C(1), \forall\, n \in \mathbb{N}, \end{aligned}$$

Hence, there exists a subsequence $\{\varphi_{1n}\}$ of $\{\varphi_n\}$ and $\varphi_{01} \in L^2(\Omega)$ such that

$$\varphi_{1n}(T_2) = \varphi(T_2; z_{1n}) \to \varphi_{01} \text{ weakly in } L^2(\Omega).$$

This together with the fact:

$$\begin{cases} (\varphi_{1n})_t(x,t) + \Delta\varphi_{1n}(x,t) - a(x,t)\varphi_{1n}(x,t) = 0 & \text{in } \Omega \times (0,T_2), \\ \varphi_{1n}(x,t) = 0 & \text{on } \partial\Omega \times (0,T_2), \\ \varphi_{1n}(x,T_2) = \varphi(T_2; z_{1n}) & \text{in } \Omega, \end{cases}$$

shows that there exists $\psi_1 \in L^q(0,T_2;L^2(\Omega)) \cap C([0,T_2-\delta];L^2(\Omega))$ for all $\delta > 0$, which satisfies

$$\begin{cases} (\psi_1)_t(x,t) + \Delta\psi_1(x,t) - a(x,t)\psi_1(x,t) = 0 & \text{in } \Omega \times (0,T_2), \\ \psi_1(x,t) = 0 & \text{on } \partial\Omega \times (0,T_2), \\ \psi_1(x,T_2) = \varphi_{01}(x) & \text{in } \Omega, \end{cases}$$

such that for all $\delta > 0$,

$$\varphi_{1n} \to \psi_1 \text{ strongly in } L^q([0,T_2];L^2(\Omega)) \cap C([0,T_2-\delta];L^2(\Omega)).$$

In particular,

$$\varphi_{1n} \to \psi_1 \text{ strongly in } L^q([0,T_2];L^2(\Omega)) \cap C([0,T_1];L^2(\Omega)), \tag{3.8}$$

and

$$\beta\varphi_{1n} \to \beta\psi_1 \text{ strongly in } L^q([0,T_2];L^2(\Omega)). \tag{3.9}$$

These together with (3.7) and (3.9) yield

$$\beta\psi_1 = \hat{\psi} \text{ in } L^q([0,T_1];L^2(\Omega)).$$

(b). Along the same way as (a), we can find a subsequence $\{\varphi_{2n}\}$ of $\{\varphi_{1n}\}$, and $\psi_2 \in L^2([0,T_3];L^2(\Omega)) \cap C([0,T_3-\delta];L^2(\Omega))$ for all $\delta > 0$, which satisfies

$$\begin{cases} (\psi_2)_t(x,t) + \Delta\psi_2(x,t) - a(x,t)\psi_2(x,t) = 0 & \text{in } \Omega \times (0,T_3), \\ \psi_2(x,t) = 0 & \text{on } \partial\Omega \times (0,T_3), \end{cases}$$

such that

$$\varphi_{2n} \to \psi_2 \text{ strongly in } L^q([0,T_3];L^2(\Omega)) \cap C([0,T_2];L^2(\Omega)).$$

This, together with (3.8), leads to

$$\psi_2|_{[0,T_1]} = \psi_1,$$

and

$$\beta\psi_2 = \hat{\psi} \text{ in } L^q([0,T_2];L^2(\Omega)).$$

(c). Similarly to (a) and (b), we can find a sequence $\{\psi_k\}$ which satisfies, for each $k \in \mathbb{N}^+$, that

- $\psi_k \in L^q([0,T_{k+1}];L^2(\Omega)) \cap C([0,T_k];L^2(\Omega))$;
- $\psi_{k+1}|_{[0,T_k]} = \psi_k$;
- ψ_k satisfies

$$\begin{cases} (\psi_k)_t(x,t) + \Delta\psi_k(x,t) - a(x,t)\psi_k(x,t) = 0 & \text{in } \Omega \times (0,T_{k+1}), \\ \psi_k(x,t) = 0 & \text{on } \partial\Omega \times (0,T_{k+1}). \end{cases}$$

- $\beta\psi_k = \hat{\psi}$ in $L^q([0,T_k];L^2(\Omega))$.

Define

$$\psi(\cdot,t) = \psi_k(\cdot,t),\ t \in [0,T_k].$$

Then, ψ is a well defined on $[0,T)$, which satisfies $\psi \in L^q([0,T];L^2(\Omega)) \cap C([0,T);L^2(\Omega))$,

$$\begin{cases} \psi_t(x,t) + \Delta\psi(x,t) - a(x,t)\psi(x,t) = 0 & \text{in } \Omega \times (0,T), \\ \psi(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

and

$$\beta\psi = \hat{\psi} = \lim_{n\to\infty} \beta\varphi(\cdot;z_n).$$

Under an isometric isomorphism, we can say $\overline{\psi} = \psi$. This complete the proof of the lemma. $\square$

We define the operator $\mathcal{T} : Y \to L^2(\Omega)$ by

$$\mathcal{T}(\varphi(\cdot;z)) = \varphi(0;z),\ \forall\ z \in L^2(\Omega), \tag{3.10}$$

which is well-defined because $Y \subset C([0,T];L^2(\Omega))$. Define the operator $\mathcal{T}_{\beta,q} : \beta\overline{Y}_{\beta,q} \to L^2(\Omega)$ by

$$\mathcal{T}_{\beta,q}(\beta\psi) = \psi(0),\ \forall\ \psi \in \overline{Y}_{\beta,q}. \tag{3.11}$$

By lemma 3.4, the operators $\mathcal{T}_{\beta,q}$ is also well-defined. In addition, it follows from the observability inequality claimed by Lemma 3.1 that the linear operator $\mathcal{T}_{\beta,q}$ is bounded.

Lemma 3.5. *If $\beta \in \mathcal{B}$ and $q \in [1,\infty)$, then the operator $\mathcal{T}_{\beta,q}$ defined by (3.11) is compact.*

Proof. By the observability inequality claimed by Lemma 3.1, it follows that the operator $\beta\overline{Y}_{\beta,q} \to L^2(\Omega)$ defined by

$$\beta\psi(\cdot,\cdot) \to \psi(\cdot,T/2),\ \forall\ \psi \in \overline{Y}_{\beta,q}$$

is bounded. Also by the property of heat equation, the operator defined by

$$\varphi(\cdot,T/2) \to \varphi(\cdot,0),\ \forall\ \varphi \in \overline{Y}_{\beta,q}$$

is compact. As a composition operator from the above two operators, $\mathcal{T}_{\beta,q}$ is compact as well. $\square$

Notice that the functional $V_q(\beta)$ in (2.3) can be written as

$$
\begin{aligned}
V_q(\beta) = & \inf_{\psi\in Y}\left[\frac{1}{2}\|\beta\psi\|^2_{L^q(0,T;L^2(\Omega))}+\langle\, y_0,\mathcal{T}(\psi)\,\rangle\right]\\
= & \inf_{\psi\in\overline{Y}_{\beta,q}}\left[\frac{1}{2}\|\beta\psi\|^2_{L^q(0,T;L^2(\Omega))}+\langle\, y_0,\mathcal{T}_{\beta,q}(\beta\psi)\,\rangle\right]\\
= & \inf_{\psi\in\overline{Y}_{\beta,q}}\left[\frac{1}{2}\|\beta\psi\|^2_{L^q(0,T;L^2(\Omega))}+\left\langle\mathcal{T}^*_{\beta,q}y_0,\beta\psi\right\rangle_{L^p(0,T;L^2(\Omega)),L^q(0,T;L^2(\Omega))}\right]\\
= & \inf_{\hat\psi\in\beta\overline{Y}_{\beta,q}}\left[\frac{1}{2}\|\hat\psi\|^2_{L^q(0,T;L^2(\Omega))}+\left\langle\mathcal{T}^*_{\beta,q}y_0,\hat\psi\right\rangle_{L^p(0,T;L^2(\Omega)),L^q(0,T;L^2(\Omega))}\right].
\end{aligned}
\tag{3.12}
$$

Let

$$\phi_{\beta,q}=\mathcal{T}^*_{\beta,q}y_0\in L^p(0,T;L^2(\Omega)).$$

We present an equivalent problem of problem $(\mathrm{Min}\,J)_{\beta,q}$ (2.3) with the extended domain:

$$(\widehat{\mathrm{Min}\,J})_{\beta,q}:\ \inf_{\zeta\in\beta\overline{Y}_{\beta,q}}\left[\frac{1}{2}\|\zeta\|^2_{L^q(0,T;L^2(\Omega))}+\langle\,\phi_{\beta,q},\zeta\,\rangle\right]. \tag{3.13}$$

The following result gives a relation between problem $(NP)_{p,\beta}$ and problem $(\widehat{\mathrm{Min}\,J})_{\beta,q}$.

Lemma 3.6. *Suppose that $\beta\in\mathcal{B}$, $y_0\in L^2(\Omega)\setminus\{0\}$, and $q\in[1,\infty)$. Then problem $(\widehat{\mathrm{Min}\,J})_{\beta,q}$ (3.13) admits a unique nonzero solution $\bar\zeta(x,t)$, and the control defined by*

$$\bar u(x,t)=\|\bar\zeta\|^{2-q}_{L^q(0,T;L^2(\Omega))}\|\bar\zeta(\cdot,t)\|^{q-2}_{L^2(\Omega)}\bar\zeta(x,t),\quad \forall\,(x,t)\in\Omega\times(0,T), \tag{3.14}$$

is an optimal control to problem $(NP)_{p,\beta}$. Moreover,

$$N_p(\beta)=\|\bar\zeta\|_{L^q(0,T;L^2(\Omega))}. \tag{3.15}$$

Proof. Since for any $q\in[1,\infty)$, by the coercive, continuity, and the strict convexity of the functional in $(\widehat{\mathrm{Min}\,J})_{\beta,q}$ (3.13), we have that the problem $(\widehat{\mathrm{Min}\,J})_{\beta,q}$ (3.13) admits a unique solution $\bar\zeta(x,t)$. We claim that

$$\bar\zeta\neq 0. \tag{3.16}$$

If this is not true, we can derive from the Euler-Lagrange equation that

$$\langle\,\phi_{\beta,q},\xi\,\rangle=0,\ \forall\,\xi\in\beta\overline{Y}_{\beta,q}.$$

It then follows from $Y_{\beta,q}\subset\overline{Y}_{\beta,q}$ that

$$\langle\, y_0,\varphi(0,z)\,\rangle=\langle\,\phi_{\beta,q},\beta\varphi(\cdot;z)\,\rangle=0,\ \forall\,z\in L^2(\Omega).$$

We claim that $\{\varphi(0,z)|z\in L^2(\Omega)\}$ is dense in $L^2(\Omega)$. Once the claim holds, the above equality implies that $y_0=0$. This contradiction shows that (3.16) is true.

Now we show that $\{\varphi(0,z)|z\in L^2(\Omega)\}$ is dense in $L^2(\Omega)$. Recalling the dual system (2.1), we define the operator L in $L^2(\Omega)$ by

$$Lz=\varphi(0,z) \text{ for any } z\in L^2(\Omega).$$

Notice that

$$\{\varphi(0,z)|z\in L^2(\Omega)\} \text{ is dense in } L^2(\Omega)\Leftrightarrow \overline{\mathcal{R}(L)}=L^2(\Omega)\Leftrightarrow \mathcal{N}(L^*)=\{0\},$$

where the last equivalence holds because of $\overline{\mathcal{R}(L)}=\mathcal{N}(L^*)^{\perp}$. For any $\hat{z}\in L^2(\Omega)$, consider the following equation

$$\begin{cases} \hat{\varphi}_t(t)-\triangle\hat{\varphi}(t)+a(t)\hat{\varphi}(t)=0,\\ \hat{\varphi}(0)=\hat{z}. \end{cases}$$

First, a direct verification shows that

$$L^*(\hat{z})=\hat{\varphi}(T).$$

By the backward uniqueness for heat equation, we have $\mathcal{N}(L^*)=\{0\}$. Second, we claim that

$$\bar{\zeta}(\cdot,t)\neq 0,\ \forall\, t\in[0,T). \tag{3.17}$$

Actually, since $\beta\overline{Y}_{\beta,q}\in C([0,T);L^2(\Omega))$, $\bar{\zeta}(\cdot,t)$ is well-defined for every $t\in[0,T)$. If there is a $t_0\in[0,T)$ such that $\bar{\zeta}(\cdot,t_0)=0$, then by Lemma 3.4, there is $\hat{\varphi}\in C([0,T);L^2(\Omega))$ which solves (3.5) such that

$$\bar{\zeta}=\beta\hat{\varphi}.$$

Since by (3.3),

$$\beta(x)\geq\sqrt{\alpha/2}\,\chi_{\Omega_1},\ \Omega_1=\left\{x\in\Omega\;\middle|\;\beta(x)\geq\sqrt{\alpha/2}\right\},\ m(\Omega_1)>0,$$

it follows that

$$\chi_{\Omega_1}\hat{\varphi}(t_0)=0.$$

By virtue of the unique continuation of heat equation ([3]), we arrive at $\hat{\varphi}(\cdot)\equiv 0$. This contradicts with (3.16), and hence (3.17) holds true.

Therefore, the control $\bar{u}(x,t)$ defined by (3.14) is well-defined and $\bar{u}(\cdot,\cdot)\in L^p(0,T;L^2(\Omega))$. Now, we show that this control is optimal to problem $(NP)_{p,\beta}$ (1.7). Since $\bar{\zeta}(x,t)$ is optimal, we can derive the corresponding Euler-Lagrange equation to the variational problem $(\widehat{\text{Min }J})_{\beta,q}$ (3.13) as follows:

$$\langle\,\bar{u},\xi\,\rangle+\langle\,\phi_{\beta,q},\xi\,\rangle=0,\ \forall\,\xi\in\beta\overline{Y}_{\beta,q}. \tag{3.18}$$

Taking $\xi=\beta\varphi(\cdot;z)$ for any $z\in L^2(\Omega)$ in (3.18), a straightforward calculation shows that

$$y(T;\beta,\bar{u})=0.$$

If $\hat{u}$ satisfies

$$y(T;\beta,\hat{u})=0, \tag{3.19}$$

we will show that

$$\|\bar{u}\|_{L^p(0,T;L^2(\Omega))} \le \|\hat{u}\|_{L^p(0,T;L^2(\Omega))}, \tag{3.20}$$

from which we see that $\bar{u}(\cdot,\cdot)$ is an optimal solution to problem $(NP)_{p,\beta}$ (1.7) and (3.15) holds.

Now, we prove (3.20). By (3.19),

$$-\langle y_0, \varphi(0;z)\rangle = \langle y(T;\beta,\hat{u}), z\rangle - \langle y_0, \varphi(0;z)\rangle = \int_0^T \langle \beta\varphi(t;z), \hat{u}(\cdot,t)\rangle dt,\ \forall\, z \in L^2(\Omega),$$

which is rewritten as

$$-\langle\, \phi_{\beta,q}, \xi\,\rangle = \langle\, \hat{u}, \xi\,\rangle,\ \forall\, \xi \in \beta Y_{\beta,q}.$$

By the density argument, it holds that

$$-\langle\, \phi_{\beta,q}, \xi\,\rangle = \langle\, \hat{u}, \xi\,\rangle,\ \forall\, \xi \in \beta \overline{Y}_{\beta,q}.$$

It then follows from (3.18) that

$$\langle\, \bar{u}, \xi\,\rangle = \langle\, \hat{u}, \xi\,\rangle,\ \forall\, \xi \in \beta \overline{Y}_{\beta,q}.$$

Taking $\xi = \bar{\zeta}$ in above quality, we have

$$\langle\, \bar{u}, \bar{\zeta}\,\rangle = \langle\, \hat{u}, \bar{\zeta}\,\rangle\,. \tag{3.21}$$

On the other hand, it follows from (3.14) that

$$\begin{aligned} \|\bar{u}\|_{L^p(0,T;L^2(\Omega))} &= \|\bar{\zeta}\|^{2-q}_{L^q(0,T;L^2(\Omega))} \left[\int_0^T \left\|\|\bar{\zeta}(\cdot,t)\|^{q-2}_{L^2(\Omega)}\bar{\zeta}(\cdot,t)\right\|^p \mathrm{d}t\right]^{\frac{1}{p}} \\ &= \|\bar{\zeta}\|^{2-q}_{L^q(0,T;L^2(\Omega))} \left[\int_0^T \|\bar{\zeta}(\cdot,t)\|^{(q-1)p}_{L^2(\Omega)} \mathrm{d}t\right]^{\frac{1}{p}} \\ &= \|\bar{\zeta}\|^{2-q}_{L^q(0,T;L^2(\Omega))} \|\bar{\zeta}\|^{\frac{q}{p}}_{L^q(0,T;L^2(\Omega))} = \|\bar{\zeta}\|_{L^q(0,T;L^2(\Omega))}. \end{aligned} \tag{3.22}$$

Hence

$$\|\bar{u}\|^2_{L^p(0,T;L^2(\Omega))} = \left\|\bar{\zeta}\right\|^2_{L^q(0,T;L^2(\Omega))} = \langle\, \bar{\zeta}, \bar{u}\,\rangle\,. \tag{3.23}$$

By (3.23), (3.21), and (3.22), we have

$$\|\bar{u}\|^2_{L^p(0,T;L^2(\Omega))} = \langle \bar{u}, \bar{\zeta}\rangle = \langle \hat{u}, \bar{\zeta}\rangle \le \|\hat{u}\| \cdot \|\bar{\zeta}\| = \|\hat{u}\| \cdot \|\bar{u}\|.$$

The result $\|\bar{u}\|_{L^p(0,T;L^2(\Omega))} \le \|\hat{u}\|_{L^p(0,T;L^2(\Omega))}$ follows immediately because $\bar{u} \neq 0$. $\square$

Remark 3.7. *By the equivalence form (3.12) for problem* $(\mathrm{Min}\, J)_{\beta,q}$ *(2.3), and the observability inequality claimed by Lemma 3.1, it is known immediately that the problem following*

$$\inf_{\psi\in\overline{Y}_{\beta,q}} \left[\frac{1}{2}\|\beta\psi\|^2_{L^q(0,T;L^2(\Omega))} + \left\langle \mathcal{T}^*_{\beta,q} y_0,\ \beta\psi\right\rangle_{L^p(0,T;L^2(\Omega)),L^q(0,T;L^2(\Omega))}\right] \tag{3.24}$$

admits a unique solution in $\overline{Y}_{\beta,q}$.

Proof of Lemma 2.1. Suppose that $\bar{\zeta}$ is an optimal solution of $(\widehat{\text{Min}\, J})_{\beta,q}$ that

$$V_q(\beta) = \frac{1}{2}\|\bar{\zeta}\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, \phi_{\beta,q}, \bar{\zeta}\,\rangle\,.$$

Replace ξ by $\bar{\zeta}$ in (3.18) to obtain

$$\langle\, \phi_{\beta,q}, \bar{\zeta}\,\rangle = -\,\langle\, \bar{u}, \bar{\zeta}\,\rangle\,.$$

This, together with (3.23), gives

$$V_q(\beta) = -\frac{1}{2}\|\bar{\zeta}\|^2_{L^q(0,T;L^2(\Omega))}.$$

The result then follows from (3.15). □

3.2 The case of two-person Stackelberg game

In this subsection, we solve the game problem (2.8). The first part presents the existence of solution to (2.8).

3.2.1 Existence of relaxed optimal actuator location

Let

$$z_q = \beta \mathcal{T}^*_{\beta,q} y_0 \in L^p(0,T;L^2(\Omega)),\ \Theta = \left\{ \theta \in L(\Omega;[0,1]) \;\Big|\; \int_\Omega \theta(x)\mathrm{d}x = \alpha \cdot \mathrm{m}(\Omega) \right\}.$$

It is clear that

$$\beta^2 \in \Theta \text{ for any } \beta \in \mathcal{B} \text{ and } \theta^{1/2} \in \mathcal{B} \;\text{ for any } \theta \in \Theta. \tag{3.25}$$

Then, the problem (GP1) (2.9) can be transformed into the following equivalent problem:

$$\begin{aligned}
&\inf_{\beta\in\mathcal{B}}\sup_{\psi\in Y}\left[-\frac{1}{2}\left(\int_0^T \left(\int_\Omega \beta^2(x)\psi^2(x,t)\mathrm{d}x \right)^{q/2} \mathrm{d}t \right)^{2/q} - \iint_{(0,T)\times\Omega} z_q(x,t)\psi(x,t)\mathrm{d}x\mathrm{d}t \right] \\
&= \inf_{\theta\in\Theta}\sup_{\psi\in Y}\left[-\frac{1}{2}\left(\int_0^T \left(\int_\Omega \theta(x)\psi^2(x,t)\mathrm{d}x \right)^{q/2} \mathrm{d}t \right)^{2/q} - \iint_{(0,T)\times\Omega} z_q(x,t)\psi(x,t)\mathrm{d}x\mathrm{d}t \right] \\
&\stackrel{\Delta}{=} \inf_{\theta\in\Theta}\sup_{\psi\in Y} F(\theta,\psi) \stackrel{\Delta}{=} \inf_{\theta\in\Theta} \hat{F}(\theta),
\end{aligned} \tag{3.26}$$

where the functional F defined on $\Theta \times Y$ by (2.11) is now given by

$$F(\theta,\psi) = -\frac{1}{2}\left(\int_0^T \left(\int_\Omega \theta(x)\psi^2(x,t)\mathrm{d}x \right)^{q/2} \mathrm{d}t \right)^{2/q} - \iint_{(0,T)\times\Omega} z_q(x,t)\psi(x,t)\mathrm{d}x\mathrm{d}t, \tag{3.27}$$

and the functional $\hat{F}$ defined on Θ is given by

$$\hat{F}(\theta) = \sup_{\psi\in Y} F(\theta,\psi),\ \forall\, \theta \in \Theta. \tag{3.28}$$

To solve problem (3.26) which is equivalent to the game problem (GP1) (2.9), we introduce the following Definitions 3.8-3.10 which can be found in Definition 38.4 on page 149 and Definition 38.5 on page 150, both in [26].

Definition 3.8. *Let Z be a Banach space and let $f : M \subseteq Z^* \to \mathbb{R} \cup \{+\infty\}$ be given. The functional f is said to be sequentially weakly* lower semi-continuous if, for any $z \in M$ and each sequence $\{z_n\} \subseteq M$ with*

$$z_n \to z \text{ weakly* in } Z^*,$$

it holds that

$$f(z) \leq \varliminf_{n\to\infty} f(z_n).$$

Definition 3.9. *Let Z be a topological space. The functional $f : M \subseteq Z \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semi-continuous if, the set*

$$M_r \triangleq \{z \in M \mid f(z) \leq r\}$$

is closed relative to M for any $r \in \mathbb{R}$.

Definition 3.10. *Let Z be a Banach space and let $f : M \subseteq Z^* \to \mathbb{R} \cup \{+\infty\}$ be given. The functional f is said to be weakly* lower semi-continuous if, the set M_r is weakly* closed in Z for any $r \in \mathbb{R}$.*

The following Propositions 3.11 is brought from Proposition 2.31 of [2, p.62]).

Proposition 3.11. *Let Z be a separable Banach space. If $f : Z^* \to \mathbb{R} \cup \{+\infty\}$ is convex, then f is weakly* lower semi-continuous if and only if f is sequentially weakly* lower semi-continuous.*

The following Proposition 3.12 comes from the fact:

$$\left\{z \in M \mid \sup_{i\in I} f_i \leq r\right\} = \bigcap_{i\in I} \{z \in M \mid f_i \leq r\}.$$

Proposition 3.12. *Let Z be a topological space and let I be an index set. If*

$$\{f_i : M \subseteq Z \to \mathbb{R} \cup \{+\infty\},\ i \in I\}$$

is a family of lower semi-continuous functionals, then $\sup_{i\in I} f_i$ is also lower semi-continuous.

The following Proposition 3.13 is actually Theorem 1.6 of [5, p.6].

Proposition 3.13. *If $r \in (0, 1]$ and*

$$f,\ g \in L^r_+ = \{f \in L^r \mid f \geq 0\},$$

then

$$\|f + g\|_{L^r} \geq \|f\|_{L^r} + \|g\|_{L^r}.$$

Now, we discuss the existence of solution to problem (3.26). To this end, let $X = L^\infty(\Omega)$ which is equipped with the weak* topology. In this way, Θ is compact in X.

Lemma 3.14. *Suppose that $q \in [1,2]$ and $y_0 \in L^2(\Omega) \setminus \{0\}$. If $\psi \in Y$ is given, then the functional $F(\cdot,\psi) :\to \mathbb{R} \cup \{+\infty\}$ defined by (3.27) is convex.*

Proof. By (3.27),

$$F(\theta,\psi) = -\frac{1}{2}\left\|\int_\Omega \theta(x)\psi^2(x,\cdot)\mathrm{d}x\right\|_{L^{\frac{q}{2}}(0,T;\mathbb{R})} - \langle z_q, \psi \rangle .$$

Notice that

$$\int_\Omega \theta(x)\psi^2(x,\cdot)\mathrm{d}x \in L^{\frac{q}{2}}_{+}(0,T),$$

where $q/2 \in (0,1]$ for $q \in [1,2]$. It then follows from Proposition 3.13 that the functional $F(\cdot,\psi)$ is convex for any $q \in [1,2]$ and $\psi \in Y$. $\square$

Lemma 3.15. *Suppose that $q \in [1,2]$ and $y_0 \in L^2(\Omega) \setminus \{0\}$. If $\psi \in Y$ is given, then the functional $F(\cdot,\psi) : \Theta \to \mathbb{R} \cup \{+\infty\}$ defined by (3.27) is sequentially weakly* lower semi-continuous.*

Proof. If there is a sequence $\{\theta_n\} \in \Theta$ such that

$$\theta_n \to \hat{\theta} \text{ weakly* in } L^\infty(\Omega),$$

then for any $\psi \in Y$ and $t \in [0,T)$,

$$\lim_{n\to\infty}\int_\Omega \theta_n(x)\psi^2(x,t)\mathrm{d}x = \int_\Omega \hat{\theta}(x)\psi^2(x,t)\mathrm{d}x \le \int_\Omega \psi^2(x,t)\mathrm{d}x.$$

Since

$$\int_0^T \left(\int_\Omega \psi^2(x,t)\mathrm{d}x\right)^{q/2}\mathrm{d}t < \infty,$$

it follows from the dominated convergence theorem, and (3.27) that

$$\lim_{n\to\infty} F(\theta_n,\psi) = F(\hat{\theta},\psi).$$

The functional $F(\cdot,\psi)$ is therefore sequentially weakly* lower semi-continuous. $\square$

Theorem 3.16. *Suppose that $q \in [1,2]$ and $y_0 \in L^2(\Omega)\setminus\{0\}$. Then the game problem (GP1) (2.9) admits a solution in Θ.*

Proof. By Lemma 3.15, the functional $F(\cdot,\psi)$ is sequentially weakly* lower semi-continuous. It follows from Proposition 3.11 and Lemma 3.14 that the functional $F(\cdot,\psi)$ is weakly* lower semi-continuous. Under the topology of X, $F(\cdot,\psi) : \Theta \subset X \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous. Furthermore, it follows from Proposition 3.12 and the definition of $\hat{F}$ in (3.28) that $\hat{F}(\cdot)$ is also lower semi-continuous. By the compactness of the domain Θ under the topology of X, there exists at least one solution to problem (3.26). Therefore, the game problem (GP1) (2.9) admits a solution in Θ. $\square$

Remark 3.17. *The set*

$$P=\left\{\beta^2 \mid \int_\Omega \beta(x)\mathrm{d}x \leq \alpha\cdot m(\Omega),\ \beta\in L(\Omega;[0,1])\right\}$$

is not weakly closed. For example, let $\Omega=(0,2)$ and $\alpha=1/4$ and take $\beta_1=\frac{1}{2}\chi_{(0,1)}$, $\beta_2=\frac{1}{2}\chi_{(1,2)}$. Then $\beta_1^2\in P$, $\beta_2^2\in P$. Consider a convex combination of β_1^2 and β_2^2: $\frac{1}{2}\beta_1^2+\frac{1}{2}\beta_2^2=1/8$ and let $\hat\beta=\dfrac{1}{2\sqrt{2}}$. Then $\hat\beta^2=1/8$. However, $\int_0^2\hat\beta(x)\mathrm{d}x=\frac{1}{\sqrt{2}}\geq\frac{1}{2}$. So $\hat\beta^2\notin P$.*

3.2.2 Value attainability of the zero-sum game

In this subsection, we will make use of the game theory to discuss the value attainability of our two-person zero-sum game (3.26). Note that for our problem (3.26),

$$V^+=\inf_{\theta\in\Theta}\sup_{\psi\in Y}F(\theta,\psi), \tag{3.29}$$

and

$$V^-=\sup_{\psi\in Y}\inf_{\theta\in\Theta}F(\theta,\psi), \tag{3.30}$$

where F is given by (3.27). It is clear that V^+ is the value of problem (3.26). Once $V^+=V^-$, we can characterize the Stackelberg equilibrium to problem (3.26) by using Proposition 2.5. To this end, we introduce an intermediate value $\hat V$ and prove successively that $V^-=\hat V$ under topological assumptions, and that $\hat V=V^+$ under convexity assumptions.

We denote by $\mathcal{K}$ all the finite subsets of Y. For any $K\in\mathcal{K}$, set

$$V_K=\inf_{\theta\in\Theta}\sup_{\psi\in K}F(\theta,\psi),\quad \hat V\triangleq\sup_{K\in\mathcal{K}}V_K=\sup_{K\in\mathcal{K}}\inf_{\theta\in\Theta}\sup_{\psi\in K}F(\theta,\psi). \tag{3.31}$$

Then

$$V^-\leq\hat V\leq V^+. \tag{3.32}$$

Lemma 3.18. *Let $q\in[1,2]$. Let V^+ and $\hat V$ be defined by (3.29) and (3.31) respectively. Then*

$$V^+=\hat V. \tag{3.33}$$

Proof. For any $K=\{\psi_1,\psi_2,\dots,\psi_n\}\in\mathcal{K}$, since from Lemma 3.15, the functional $F(\cdot,\psi_j)$ is sequentially weakly* lower semi-continuous in Θ for any $j\in\{1,2,\dots,n\}$, it follows from the proof of Theorem 3.16 that there is $\theta_K\in\Theta$ such that

$$\sup_{\psi\in K}F(\theta_K,\psi)=\inf_{\theta\in\Theta}\sup_{\psi\in K}F(\theta,\psi).$$

This together with the definition of $\hat V$ enables us to derive

$$F(\theta_K,\psi)\leq\sup_{\hat\psi\in K}F(\theta_K,\hat\psi)=\inf_{\theta\in\Theta}\sup_{\hat\psi\in K}F(\theta,\hat\psi)\leq\sup_{\hat K\in\mathcal{K}}\inf_{\theta\in\Theta}\sup_{\hat\psi\in\hat K}F(\theta,\hat\psi)=\hat V,\quad\forall\,\psi\in K. \tag{3.34}$$

For any $\psi\in Y$, denote

$$S_\psi\triangleq\left\{\theta\in\Theta \mid F(\theta,\psi)\leq\hat V\right\}.$$

It follows from (3.34) that the set S_ψ is nonempty and

$$\{\theta_K\} \subset \bigcap_{\psi \in K} S_\psi \neq \emptyset. \tag{3.35}$$

In addition, since $F(\cdot,\psi)$ is weakly* lower semi-continuous, S_ψ is weakly* closed in $L^\infty(\Omega)$. In other words, S_ψ is closed under the topology of X. This, together with (3.35), implies that

the intersection of any finite subsets of $\{S_\psi, \psi \in Y\}$ is nonempty.

By the compactness of Θ,

$$\bigcap_{\psi \in Y} S_\psi \neq \emptyset.$$

Hence, there is $\bar{\theta}$ such that

$$\sup_{\psi \in Y} F(\bar{\theta},\psi) \le \hat{V},$$

and so

$$\inf_{\theta \in \Theta} \sup_{\psi \in Y} F(\theta,\psi) \le \hat{V}.$$

This, together with (3.32), completes the proof of the lemma. □

The following Proposition 3.19 is Proposition 8.3 of [1, p.132].

Proposition 3.19. *Let $\hat{E}$ and $\hat{F}$ be two convex sets and let the function $f(\cdot,\cdot)$ be defined in $\hat{E} \times \hat{F}$. Let $\mathcal{F}$ be the set of all finite subsets of $\hat{F}$ and*

$$\hat{V} = \sup_{K \in \mathcal{F}} \inf_{x \in \hat{E}} \sup_{\psi \in K} f(x,y), \ \ V^- = \sup_{y \in \hat{F}} \inf_{x \in \hat{E}} f(x,y).$$

Suppose that a) for any $y \in \hat{F}$, $x \to f(x,y)$ is convex; and b) for any $x \in \hat{E}$, $x \to f(x,y)$ is concave. Then $\hat{V} = V^-$.

Lemma 3.20. *Let $q \in [1,2]$ and let V^- and $\hat{V}$ be defined by (3.30) and (3.31), respectively. Then*

$$\hat{V} = V^-. \tag{3.36}$$

Proof. It is clear that both Θ and Y are convex. Let $\theta \in \Theta$ and let $\beta \in \mathcal{B}$ be such that $\beta^2 = \theta$. Since by (3.27),

$$F(\theta,\psi) = -\frac{1}{2}\|\beta\psi\|^2_{L^q(0,T;L^2(\Omega))} - \langle\, z_q, \psi \,\rangle,$$

and

$$\left\|\frac{\psi_1+\psi_2}{2}\right\|^2_{L^q(0,T;L^2(\Omega))} \le \frac{1}{2}\,\|\psi_1\|^2_{L^q(0,T;L^2(\Omega))} + \frac{1}{2}\,\|\psi_2\|^2_{L^q(0,T;L^2(\Omega))}\,,\ \forall\, \psi_1,\psi_2 \in L^q(0,T;L^2(\Omega)),$$

the functional $F(\theta,\cdot)$ is concave for any $q \in [1,\infty)$ and $\theta \in \Theta$. On the other hand, it follows from Lemma 3.14 that the functional $F(\cdot,\psi)$ is convex for any $q \in [1,2]$ and $\psi \in Y$. Apply Proposition 3.19 to obtain (3.36). This completes the proof of the lemma. □

Combining the above results, we have proved the following Theorem 3.21.

Theorem 3.21. *Suppose that $q \in [1,2]$ and let V^+ and V^- be defined by (3.29) and (3.30) respectively. Then*

$$V^- = V^+. \tag{3.37}$$

Remark 3.22. *In the original problem, there are two important cases. One is $p = 2$, and the other is $p = \infty$. Their corresponding conjugate exponents are $q = 2$ and $q = 1$ respectively. It is fortunate that Theorem 3.21 is valid for both these cases.*

3.2.3 Nash equilibrium

The value attainability for a given two-person zero-sum game is a necessary condition to the existence of the Nash equilibriums. To discuss further about the solution to the Stackleberg game problem (GP1) (2.9) or equivalently problem (3.29), we need to discuss another Stackleberg game problem (3.30), in other words, we should discuss the following problem:

$$\inf_{\psi\in Y}\sup_{\theta\in\Theta}\left[\frac{1}{2}\left(\int_0^T\left(\int_\Omega \theta(x)\psi(x,t)^2 \mathrm{d}t\right)^{\frac{q}{2}}\mathrm{d}t\right)^{\frac{2}{q}} + \langle\, y_0, \psi(0)\,\rangle\right]. \tag{3.38}$$

Define a non-negative nonlinear functional on Y by

$$NF(\psi) = \sup_{\theta\in\Theta}\left(\int_0^T\left(\int_\Omega \theta(x)\psi(x,t)^2 \mathrm{d}t\right)^{\frac{q}{2}}\mathrm{d}t\right)^{\frac{1}{q}},\ \forall\, \psi\in Y. \tag{3.39}$$

Lemma 3.23. *For $q \in [1, +\infty)$, the functional $NF(\cdot)$ defined by (3.39) is a norm for the space Y defined by (2.5).*

Proof. It is clear that

$$NF(\psi) \geq 0,\ \forall\, \psi \in Y \text{ and } \psi = 0 \Rightarrow NF(\psi) = 0.$$

By (3.25),

$$NF(\psi) = \sup_{\beta\in\mathcal{B}} \|\beta\psi\|_{L^q(0,T;L^2(\Omega))}.$$

Furthermore, if $NF(\psi) = 0$, then $\beta\psi = 0$ for any $\beta \in \mathcal{B}$. By

$$\chi_{\{x\in\Omega|\beta(x)\geq\sqrt{\alpha/2}\}}|\psi(x,t)| \leq \frac{\beta(x)}{\sqrt{\alpha/2}}|\psi(x,t)|,$$

we have

$$\chi_{\{x\in\Omega|\beta(x)\geq\sqrt{\alpha/2}\}}\psi = 0.$$

It then follows from (3.3) and the unique continuation for heat equation ([3]) that $\psi = 0$. Therefore, $NF(\psi) = 0$ if and only if $\psi = 0$. Finally, a direct computation shows that

$$NF(c\psi) = |c|NF(\psi), \forall\, \psi \in Y,\ c \in \mathbb{R}.$$

By

$$\|\beta(\psi_1+\psi_2)\|_{L^q(0,T;L^2(\Omega))} \le \|\beta\psi_1\|_{L^q(0,T;L^2(\Omega))} + \|\beta\psi_2\|_{L^q(0,T;L^2(\Omega))}, \forall\, \beta^2=\theta\in\Theta,$$

we have

$$\begin{aligned}&\left(\int_0^T\left(\int_\Omega \theta(x)(\psi_1(x,t)+\psi_2(x,t))^2 \mathrm{d}t\right)^{\frac{q}{2}}\mathrm{d}t\right)^{\frac{1}{q}}\\ &\le \left(\int_0^T\left(\int_\Omega \theta(x)\psi_1(x,t)^2\mathrm{d}t\right)^{\frac{q}{2}}\mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_0^T\left(\int_\Omega \theta(x)\psi_2(x,t)^2\mathrm{d}t\right)^{\frac{q}{2}}\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

So,

$$N\!F(\psi_1+\psi_2)\le N\!F(\psi_1)+N\!F(\psi_2).$$

This shows that $N\!F$ is a norm for the space Y. □

Definition 3.24. *Owing to Lemma 3.23, we can denote the norm given by the functional $N\!F(\cdot)$ as $\|\cdot\|_{N\!F}$. It is clear that the space $(Y,\ \|\cdot\|_{N\!F})$ is a normed linear space. We set $\left(\overline{Y_q},\ \|\cdot\|_{N\!F}\right)$ as the completion space of $(Y,\ \|\cdot\|_{N\!F})$.*

Along the same line in the proof of Lemma 3.4, we have the following Lemma 3.25.

Lemma 3.25. *Let $1\le q<\infty$. Then under an isometric isomorphism, any element of $\overline{Y_q}$ can be expressed as a function $\hat\varphi\in C([0,T);L^2(\Omega))$ which satisfies (in the sense of weak solution)*

$$\begin{cases}\hat\varphi_t(x,t)+\Delta\hat\varphi(x,t)-a(x,t)\hat\varphi(x,t)=0 & \text{in } \Omega\times(0,T),\\ \hat\varphi(x,t)=0 & \text{on } \partial\Omega\times(0,T),\end{cases}$$

and $N\!F(\hat\varphi)=\lim\limits_{n\to\infty}N\!F(\varphi(\cdot;z_n))$ for some sequence $\{z_n\}\subset L^2(\Omega)$, where $\varphi(\cdot;z_n)$ is the solution of (2.1) with initial value $z=z_n$.

Remark 3.26. *By Lemma 3.25, we have the following inclusion:*

$$\overline{Y_q}\subseteq L^q(0,T;L^2(\Omega)). \tag{3.40}$$

Indeed, suppose that $n_0\in\mathbb{N}$ so that $n_0\ge 1/\alpha$. There are n_0 measurable subsets $\omega_1,\omega_2,\dots,\omega_{n_0}$ of Ω such that

$$\omega_j\in\mathcal{W},\quad \forall\, j\in\{1,2,\dots,n_0\},\quad \bigcup_{j=1}^{n_0}\omega_j=\Omega.$$

The inclusion (3.40) then follows from

$$\begin{aligned}&\int_0^T\left(\int_\Omega\psi(x,t)^2\mathrm{d}x\right)^{\frac{q}{2}}\mathrm{d}t\le\int_0^T\left(\sum_{j=1}^{n_0}\int_\Omega\chi_{\omega_j}(x)\psi(x,t)^2\mathrm{d}x\right)^{\frac{q}{2}}\mathrm{d}t\\ &\le\int_0^T\left[n_0\sum_{j=1}^{n_0}\left(\int_\Omega\chi_{\omega_j}(x)\psi(x,t)^2\mathrm{d}x\right)^q\right]^{1/2}\mathrm{d}t\\ &\le\sqrt{n_0}\int_0^T\sum_{j=1}^{n_0}\left(\int_\Omega\chi_{\omega_j}(x)\psi(x,t)^2\mathrm{d}x\right)^{q/2}\mathrm{d}t\end{aligned} \tag{3.41}$$

$$\leq \sqrt{n_0} \sum_{j=1}^{n_0} \|\psi\|_{NF}^{q} = n_0^{3/2} \|\psi\|_{NF}^{q},$$

where the Schwartz's inequality is used in the second inequality of (3.41) and the last inequality in (3.41) is derived from $\sqrt{\sum a_i} \leq \sum \sqrt{a_i}$.

Furthermore, for any $\beta \in \mathcal{B}$*, it follows from*

$$\|\beta\psi\|_{L^2(0,T:L^2(\Omega))} \leq NF(\psi),\ \forall\ \psi \in Y.$$

that

$$\overline{Y_q} \subseteq \overline{Y_{\beta,q}}, \quad \forall\ \beta \in \mathcal{B}. \tag{3.42}$$

Recalling that Y *is dense in* $\overline{Y_{\beta,q}}$ *and* $\sup\limits_{\psi \in Y} F(\theta,\psi) = \sup\limits_{\psi \in \overline{Y_{\beta,q}}} F(\theta,\psi)$ *with* $\theta = \beta^2$*, we have*

$$\sup_{\psi \in Y} F(\theta,\psi) = \sup_{\psi \in \overline{Y_q}} F(\theta,\psi) = \sup_{\psi \in \overline{Y_{\beta,q}}} F(\theta,\psi). \tag{3.43}$$

Now, we discuss the following extended game problem of (3.38):

$$\begin{aligned}
(GP2):\quad & \inf_{\psi \in \overline{Y_q}} \sup_{\theta \in \Theta} \left[\frac{1}{2} \left(\int_0^T \left(\int_\Omega \theta(x)\psi(x,t)^2 \mathrm{d}x \right)^{\frac{q}{2}} \mathrm{d}t \right)^{\frac{2}{q}} + \langle\, y_0, \psi(0)\, \rangle \right] \\
& = \inf_{\psi \in \overline{Y_q}} \left[\frac{1}{2} \|\psi\|_{NF}^2 + \langle\, y_0, \psi(0)\, \rangle \right].
\end{aligned} \tag{3.44}$$

Notice that the functional in problem (GP2) (3.44) is strictly convex, coercive, and continuous. Similarly to Lemma 3.6, we have the following Lemma 3.27.

Lemma 3.27. *For any* $y_0 \in L^2(\Omega) \setminus \{0\}$ *and* $q \in [1,\infty)$*, the game problem (GP2) (3.44) admits a unique nonzero solution.*

Now we present Nash equilibrium problem of two-person zero-sum game:

$$(GP3):\quad \text{To find } \bar{\theta} \in \Theta,\ \bar{\psi} \in \overline{Y_q} \text{ such that } F(\bar{\theta},\bar{\psi}) = \sup_{\psi \in \overline{Y_q}} F(\bar{\theta},\psi) = \inf_{\theta \in \Theta} F(\theta,\bar{\psi}), \tag{3.45}$$

where $F(\theta,\psi)$ is defined by (3.27). The following Theorem 3.28 is about the existence of the Nash equilibrium to the two-person zero-sum game problem (GP3) (3.45).

Theorem 3.28. *Let* $q \in [1,2]$ *and let* $\bar{\psi}$ *be a solution to problem (GP2) (3.44). Then problem (GP3) (3.45) admits at least one Nash equilibrium. Furthermore, if* $\bar{\beta}$ *is a relaxed optimal actuator location to problem (1.8), then* $(\bar{\theta} = \bar{\beta}^2, \bar{\psi})$ *is a Nash equilibrium to problem (GP3) (3.45). Conversely, if* $(\hat{\theta},\hat{\psi})$ *is a Nash equilibrium of problem (GP3) (3.45), then* $\hat{\psi} = \bar{\psi}$*, and* $\hat{\beta} = \hat{\theta}^{1/2}$ *is a relaxed optimal actuator location to problem (1.8).*

Proof. In terms of (3.43),

$$V^+ = \inf_{\theta\in\Theta}\sup_{\psi\in Y} F(\theta,\psi) = \inf_{\theta\in\Theta}\sup_{\psi\in\overline{Y_q}} F(\theta,\psi). \tag{3.46}$$

Notice that

$$V^- = \sup_{\psi\in Y}\inf_{\theta\in\Theta} F(\theta,\psi) \le \sup_{\psi\in\overline{Y_q}}\inf_{\theta\in\Theta} F(\theta,\psi) \le \inf_{\theta\in\Theta}\sup_{\psi\in\overline{Y_q}} F(\theta,\psi).$$

It follows from Theorem 3.21 that

$$\inf_{\theta\in\Theta}\sup_{\psi\in\overline{Y_q}} F(\theta,\psi) = \sup_{\psi\in\overline{Y_q}}\inf_{\theta\in\Theta} F(\theta,\psi). \tag{3.47}$$

Furthermore, by (3.46) and (3.25),

$$\begin{aligned}
&\text{if } \bar\beta \text{ is a solution to problem (GP1) (2.9), then } \bar\theta \text{ is a solution to } \inf_{\theta\in\Theta}\sup_{\psi\in\overline{Y_q}} F(\theta,\psi);\\
&\text{if } \bar\theta \text{ is a solution to} \inf_{\theta\in\Theta}\sup_{\psi\in\overline{Y_q}} F(\theta,\psi), \text{ then } \bar\beta \text{ is a solution to problem (GP1) (2.9),}
\end{aligned} \tag{3.48}$$

where $\bar\theta = \bar\beta^2$. Recalling Proposition 2.5, we have the following results:

- Equation (3.47) ensures that problem (GP3) attains its value;
- Problem $(GP2)$ (3.44) admits a unique solution $\bar\psi$ by Lemma 3.27;
- Problem (GP1) (2.9) admits a solution by Theorem 3.16 and (3.48).

It follows from Proposition 2.5 that problem (GP3) admits at least one Nash equilibrium. Furthermore, if $\bar\theta$ is a solution to $\inf_{\theta\in\Theta}\sup_{\psi\in\overline{Y_q}} F(\theta,\psi)$, then $(\bar\theta,\bar\psi)$ is a Nash equilibrium to problem (GP3). Conversely, if $(\hat\theta,\hat\psi)$ is a Nash equilibrium of problem (GP3), then $\hat\theta$ is a solution to problem $\inf_{\theta\in\Theta}\sup_{\psi\in\overline{Y_q}} F(\theta,\psi)$ and $\hat\psi$ solves $\sup_{\psi\in\overline{Y_q}}\inf_{\theta\in\Theta} F(\theta,\psi)$. By the uniqueness from Lemma 3.27, it holds that $\hat\psi = \bar\psi$. That, together with (3.48) and the equivalence between problem (1.8) and problem (GP1), Theorem 3.28 is derived directly. $\square$

Proof of Theorem 1.1. If $p \in [2,+\infty]$, then $q \in [1,2]$ and vice verse. Notice that $(\hat\theta,\hat\psi)$ is a Nash equilibrium of problem (GP3) if and only if $(\hat\beta,\hat\psi)$ is a Nash equilibrium of problem (1.10), where $\hat\beta^2 = \hat\theta$. By a direct verification, Theorem 1.1 follows from Theorem 3.28. $\square$

Remark 3.29. *For any Nash equilibrium to problem (GP3) in Theorem 3.28, the second component $\bar\psi$ is the unique solution to problem (GP2) (Lemma 3.27). Thus for any solution $\bar\beta$ to problem (1.8), the set $\left\{\hat\psi \in \overline{Y_q} \mid (\bar\beta,\hat\psi) \text{ is a Nash equilibrium}\right\}$ defined in (1.11) is a singleton and independent of $\bar\beta$. Thus for any solution $\bar\beta$ to problem (1.8), $(\bar\beta,\bar\psi)$ is a Nash equilibrium of problem (1.10). By the definition of Nash equilibrium, $\bar\beta$ solves the following problem:*

$$\sup_{\beta\in\mathcal{B}}\left[\frac{1}{2}\left\|\beta\bar\psi(\cdot)\right\|^2_{L^q(0,T;L^2(\Omega))} + \langle\, y_0, \bar\psi(0)\,\rangle\right].$$

Equivalently, $\bar{\beta}$ solves problem (1.12). We thus have all results of Remark 1.2.

If $\bar{\beta}$ is a relaxed optimal actuator location, then $(\bar{\beta}, \bar{\psi})$ is a Nash equilibrium of problem (1.10). So $\bar{\beta}$ is optimal for this fixed $\bar{\psi}$. However, if there is $\hat{\beta}$ such that $\hat{\beta}$ is optimal for the fixed $\bar{\psi}$, we can not derive that $\bar{\psi}$ is also optimal for this $\hat{\beta}$. Therefore, we can not say that $\hat{\beta}$ is also a relaxed optimal actuator location. This implies that the condition in Remark 1.2 is only a necessary condition.

3.3 Optimal actuator location for the case of $p = 2$

Though we have derived the existence for the relaxation problem, the existence of the optimal actuator location to the classical problem (1.4) is still not known. A key problem leading the relaxation solution to the existence of the classical problem (1.4) is whether the following equality holds:

$$\inf_{\beta \in \mathcal{B}} N_p(\beta) = \inf_{\omega \in \mathcal{W}} N_p(\omega)? \tag{3.49}$$

To establish this equality, we need to learn more about the optimal a relaxed actuator location $\bar{\beta}$. Recall Remark 1.2 that if $\bar{\beta}$ is relaxed actuator location, then $\bar{\beta}$ solves problem (1.12). Thus $\bar{\theta} = \bar{\beta}^2$ solves

$$\sup_{\theta \in \Theta} \left(\int_0^T \left(\int_\Omega \theta(x) \bar{\psi}(x,t)^2 \mathrm{d}t \right)^{\frac{q}{2}} \mathrm{d}t \right)^{\frac{2}{q}}.$$

That is to say,

$$\int_0^T \left(\int_\Omega \bar{\theta}(x) \bar{\psi}(x,t)^2 \mathrm{d}t \right)^{\frac{q}{2}} \mathrm{d}t = \sup_{\theta \in \Theta} \int_0^T \left(\int_\Omega \theta(x) \bar{\psi}(x,t)^2 \mathrm{d}t \right)^{\frac{q}{2}} \mathrm{d}t. \tag{3.50}$$

In this subsection, we limit ourselves to the case of $p = 2$. We show that when $p = 2$, the equality (3.49) is indeed valid, which relies on the fact that the integration orders in equation (3.50) with respect to the variables t and x can be exchanged.

First of all, we present a preliminary result about the following problem

$$\sup_{\theta \in \Theta} \int_\Omega \theta(x) \phi(x) \mathrm{d}x, \tag{3.51}$$

where $\phi(\cdot) \in L^1(\Omega)$. To this purpose, we define, for any $\phi \in L^1(\Omega)$ and $c \in \mathbb{R}$, that

$$\begin{aligned} &\Omega[\phi \geq c] = \left\{ x \in \Omega \mid \phi(x) \geq c \right\}, \quad \Omega[\phi = c] = \left\{ x \in \Omega \mid \phi(x) = c \right\}, \\ &\Omega[\phi > c] = \left\{ x \in \Omega \mid \phi(x) > c \right\}, \quad \Omega[\phi < c] = \left\{ x \in \Omega \mid \phi(x) < c \right\}. \end{aligned} \tag{3.52}$$

Let

$$M_\phi(c) = m(\Omega[\phi \geq c]) \text{ for any } \phi \in L^1(\Omega) \text{ and } c \in \mathbb{R}. \tag{3.53}$$

It is clear that the function $M_\phi(c)$ is monotone decreasing with respect to c. By

$$\lim_{\varepsilon \to 0+} \Omega[\phi \geq c - \varepsilon] = \bigcap_{\varepsilon > 0} \Omega[\phi \geq c - \varepsilon] = \Omega[\phi \geq c],$$

we have

$$\lim_{\varepsilon\to 0+} M_\phi(c-\varepsilon) = M_\phi(c),\ \forall\, c\in\mathbb{R}. \tag{3.54}$$

This shows that $M_\phi(\cdot)$ is continuous from the left for any given $\phi\in L^1(\Omega)$. Since

$$\lim_{c\to+\infty} M_\phi(c) = 0, \quad \lim_{c\to-\infty} M_\phi(c) = m(\Omega),$$

the real c_ϕ given by

$$c_\phi = \max\left\{c\in\mathbb{R}\;\middle|\; M_\phi(c)\ge \alpha\cdot m(\Omega)\right\}, \tag{3.55}$$

is well-defined. Hence

$$M_\phi(c_\phi)\ge \alpha\cdot m(\Omega)\ge M_\phi(c_\phi+)\stackrel{\Delta}{=}\lim_{\varepsilon\to 0+} M_\phi(c_\phi+\varepsilon), \tag{3.56}$$

and

$$M_\phi(c_\phi+\varepsilon) < \alpha m(\Omega),\ \forall\,\varepsilon>0. \tag{3.57}$$

Let

$$\bar{\alpha}_\phi \stackrel{\Delta}{=} \frac{M_\phi(c_\phi)}{m(\Omega)}, \quad \underline{\alpha}_\phi \stackrel{\Delta}{=} \frac{M_\phi(c_\phi+)}{m(\Omega)}. \tag{3.58}$$

It follows from (3.56) that

$$\bar{\alpha}_\phi \ge \alpha \ge \underline{\alpha}_\phi. \tag{3.59}$$

Since

$$\lim_{\varepsilon\to 0+} \Omega[\phi\ge c+\varepsilon] = \bigcup_{\varepsilon>0} \Omega[\phi\ge c+\varepsilon] = \Omega[\phi>c],$$

it follows that

$$M_\phi(c_\phi+) = m(\Omega[\phi>c_\phi]).$$

By the definition of $\underline{\alpha}_\phi$ in (3.58),

$$m(\Omega[\phi>c_\phi]) = \underline{\alpha}_\phi\cdot m(\Omega). \tag{3.60}$$

This, together with (3.58) and (3.59), implies that

$$m(\Omega[\phi=c_\phi]) = (\bar{\alpha}_\phi-\underline{\alpha}_\phi)m(\Omega)\ge(\alpha-\underline{\alpha}_\phi)m(\Omega). \tag{3.61}$$

The following result is about problem (3.51).

Lemma 3.30. *Let $\phi(\cdot)\in L^1(\Omega)$. Then the problem (3.51) admits a solution $\bar{\theta}(x)=\chi_\omega(x)\in\mathcal{W}$. Moreover, the function $\bar{\theta}(\cdot)\in\Theta$ is a solution to problem (3.51) if and only if it satisfies the following two conditions*

$$\bar{\theta}(x)=1,\forall\, x\in\Omega[\phi>c_\phi]\ \textit{a.e. and}\ \ \bar{\theta}(x)=0,\forall\, x\in\Omega[\phi<c_\phi]\ \textit{a.e.} \tag{3.62}$$

where c_ϕ is defined by (3.55). As a consequence, the problem (3.51) admits a slolution $\chi_\omega\in\mathcal{W}$ if and only if $\bar{\theta}(x)=\chi_\omega(x)$ satisfies (3.62).

Proof. For any $\theta(\cdot) \in \Theta$, it holds that

$$
\begin{aligned}
& \int_\Omega \theta(x)\phi(x)\mathrm{d}x \\
= & \int_{\Omega[\phi>c_\phi]} \theta(x)\phi(x)\mathrm{d}x + c_\phi \int_{\Omega[\phi=c_\phi]} \theta(x)\mathrm{d}x + \int_{\Omega[\phi<c_\phi]} \theta(x)\phi(x)\mathrm{d}x \\
= & \int_{\Omega[\phi>c_\phi]} \phi(x)\mathrm{d}x - \int_{\Omega[\phi>c_\phi]} (1-\theta(x))\phi(x)\mathrm{d}x + c_\phi \int_{\Omega[\phi=c_\phi]} \theta(x)\mathrm{d}x + \int_{\Omega[\phi<c_\phi]} \theta(x)\phi(x)\mathrm{d}x \\
\le & \int_{\Omega[\phi>c_\phi]} \phi(x)\mathrm{d}x - \int_{\Omega[\phi>c_\phi]} (1-\theta(x))c_\phi\mathrm{d}x + c_\phi \int_{\Omega[\phi=c_\phi]} \theta(x)\mathrm{d}x + \int_{\Omega[\phi<c_\phi]} \theta(x)\phi(x)\mathrm{d}x \\
= & \int_{\Omega[\phi>c_\phi]} \phi(x)\mathrm{d}x - \underline{\alpha}_\phi m(\Omega)\cdot c_\phi + c_\phi \int_{\Omega[\phi\ge c_\phi]} \theta(x)\mathrm{d}x + \int_{\Omega[\phi<c_\phi]} \theta(x)\phi(x)\mathrm{d}x \\
\le & \int_{\Omega[\phi>c_\phi]} \phi(x)\mathrm{d}x - \underline{\alpha}_\phi m(\Omega)\cdot c_\phi + c_\phi \int_{\Omega[\phi\ge c_\phi]} \theta(x)\mathrm{d}x + c_\phi \int_{\Omega[\phi<c_\phi]} \theta(x)\mathrm{d}x \\
= & \int_{\Omega[\phi>c_\phi]} \phi(x)\mathrm{d}x - \underline{\alpha}_\phi m(\Omega)\cdot c_\phi + c_\phi \int_{\Omega} \theta(x)\mathrm{d}x \\
= & \int_{\Omega[\phi>c_\phi]} \phi(x)\mathrm{d}x + (\alpha - \underline{\alpha}_\phi) m(\Omega)\cdot c_\phi
\end{aligned}
\tag{3.63}
$$

In (3.63), the third equation comes from (3.60). Hence

$$
\sup_{\theta\in\Theta} \int_\Omega \theta(x)\phi(x)\mathrm{d}x \le \int_{\Omega[\phi>c_\phi]} \phi(x)\mathrm{d}x + (\alpha - \underline{\alpha}_\phi) m(\Omega)\cdot c_\phi. \tag{3.64}
$$

If $\bar\theta \in \Theta$ and (3.62) holds, then it follows from (3.60) that

$$
\int_{\Omega[\phi=c_\phi]} \bar\theta(x)\mathrm{d}x = (\alpha - \underline{\alpha}_\phi) m(\Omega). \tag{3.65}
$$

This, together with (3.62), implies that

$$
\int_{\Omega[\phi>c_\phi]} \phi(x)\mathrm{d}x + (\alpha - \underline{\alpha}_\phi) m(\Omega)\cdot c_\phi = \int_\Omega \bar\theta(x)\phi(x)\mathrm{d}x.
$$

Thus $\bar\theta$ is a solution and

$$
\max_{\theta\in\Theta} \int_\Omega \theta(x)\phi(x)\mathrm{d}x = \int_{\Omega[\phi>c_\phi]} \phi(x)\mathrm{d}x + (\alpha - \underline{\alpha}_\phi) m(\Omega)\cdot c_\phi. \tag{3.66}
$$

For each measurable subset E of $\Omega[\phi = c_\phi]$ with

$$
m(E) = (\alpha - \underline{\alpha}_\phi) m(\Omega),
$$

define

$$
\hat\theta = \chi_{\Omega[\phi>c_\phi]\cup E}.
$$

A direct computation shows that $\hat\theta \in \mathcal{W}$ and (3.62) holds. Thus problem (3.51) admits a solution in $\mathcal{W}$. On the other hand, if $\hat\theta$ is a solution, we can derive (3.62) by (3.66) directly. This completes the proof of the lemma. $\square$

Remark 3.31. *Define a set-valued operator $\mathcal{O} : L^1(\Omega) \to 2^\Theta$ as follows:*

$$\text{For any } \phi \in L^1(\Omega), \theta \in \mathcal{O}(\phi) \text{ if and only if } \theta \in \Theta \text{ and condition (3.62) holds.} \tag{3.67}$$

By Lemma 3.30, it is easy to verify that θ solves problem (3.51) if and only if $\theta \in \mathcal{O}(\phi)$, in other words,

$$\mathcal{O}(\phi) \text{ is the solution set to problem (3.51).} \tag{3.68}$$

Now we discuss the game problem (GP2) (3.44) for $p = q = 2$, that is,

$$(GP4): \quad \begin{aligned} &\inf_{\psi\in\overline{Y_2}} \sup_{\theta\in\Theta} \left[\frac{1}{2}\int_0^T \int_\Omega \theta(x)|\psi(x,t)|^2 \mathrm{d}x\mathrm{d}t + \langle\, y_0, \psi(0)\,\rangle\right] \\ &= \inf_{\psi\in\overline{Y_2}} \max_{\theta\in\Theta} \left[\frac{1}{2}\langle\, \theta, G_\psi\,\rangle + \langle\, y_0, \psi(0)\,\rangle\right], \end{aligned} \tag{3.69}$$

where the operator $G : L^2(\Omega \times (0,T)) \to L^1(\Omega)$ is defined by

$$G_\psi(x) = \int_0^T |\psi(x,t)|^2 \mathrm{d}t,\ x \in \Omega \text{ a.e.} \tag{3.70}$$

by (3.40), G is well-defined in the space $\overline{Y_2}$.

Proposition 3.32. *Let the operator G and the set-valued operator $\mathcal{O}$ be defined by (3.70) and (3.67), respectively. If $\bar{\psi}$ is a solution to (GP4) (3.69), then $\hat{\beta} \in \mathcal{B}$ is a solution to problem (1.8) if and only if $\hat{\theta} = \hat{\beta}^2 \in \Theta$ solves problem (3.51) with $\phi = G_{\bar{\psi}}$, i.e.*

$$\hat{\theta} \in \mathcal{O}(G_{\bar{\psi}}). \tag{3.71}$$

Proof. The necessity follows from Theorem 1.1. For the sufficiency, we suppose (3.71). The remaining proof will be split into two steps.

Step 1. Define a nonlinear functional $\mathcal{F}$ in $L^1(\Omega)$ by

$$\mathcal{F}(g) = \frac{1}{2}\max_{\theta\in\Theta} \int_\Omega \theta(x)g(x)\mathrm{d}x,\ \forall\, g \in L^1(\Omega).$$

Then, we can rewrite problem (GP4) (3.69) as the following problem:

$$\inf_{\psi\in\overline{Y_q}} \left(\mathcal{F}(G_\psi) + \langle\, y_0, \psi(0)\,\rangle\right). \tag{3.72}$$

Since $\bar{\psi}$ is a solution to problem (3.72),

$$\lim_{\varepsilon\to0+} \frac{1}{\varepsilon}\left[\mathcal{F}(G_{\varepsilon\psi+\bar{\psi}}) + \langle\, y_0, (\varepsilon\psi+\bar{\psi})(0)\,\rangle - \mathcal{F}(G_{\bar{\psi}}) - \langle\, y_0, \bar{\psi}(0)\,\rangle\right] = 0, \forall\, \psi \in \overline{Y_2}. \tag{3.73}$$

Denote

$$\bar{f} = \int_0^T \bar{\psi}(\cdot,t)^2 \mathrm{d}t = G_{\bar{\psi}},\ \text{and } f_\psi = \int_0^T \bar{\psi}(\cdot,t)\psi(\cdot,t)\mathrm{d}t, \forall\, \psi \in \overline{Y_2}. \tag{3.74}$$

Then for any $\psi \in \overline{Y_2}$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left[\mathcal{F}(G_{\varepsilon\psi+\bar\psi})-\mathcal{F}(G_{\bar\psi})\right]\\
=\;&\lim_{\varepsilon\to 0+}\frac{1}{2\varepsilon}\Bigg[\max_{\theta\in\Theta}\int_\Omega\theta(x)\mathrm{d}x\int_0^T\left[|\bar\psi(x,t)|^2+2\varepsilon\bar\psi(x,t)\psi(x,t)+\varepsilon^2|\psi(x,t)|^2\right]\mathrm{d}t\\
&\qquad-\max_{\theta\in\Theta}\int_\Omega\theta(x)\mathrm{d}x\int_0^T|\bar\psi(x,t)|^2\mathrm{d}t\Bigg]\\
=\;&\lim_{\varepsilon\to 0+}\frac{1}{2\varepsilon}\Bigg[\max_{\theta\in\Theta}\int_\Omega\theta(x)\mathrm{d}x\left[\int_0^T|\bar\psi(x,t)|^2\mathrm{d}t+2\varepsilon\int_0^T\bar\psi(x,t)\psi(x,t)\mathrm{d}t\right]\\
&\qquad-\max_{\theta\in\Theta}\int_\Omega\theta(x)\mathrm{d}x\int_0^T|\bar\psi(x,t)|^2\mathrm{d}t\Bigg]\\
=\;&\lim_{\varepsilon\to 0+}\frac{1}{2\varepsilon}\left[\max_{\theta\in\Theta}\int_\Omega\theta(x)\left[\bar f(x)+2\varepsilon f_\psi(x)\right]\mathrm{d}x-\max_{\theta\in\Theta}\int_\Omega\theta(x)\bar f(x)\mathrm{d}x\right]\\
=\;&\int_{\Omega[\bar f>c_{\bar f}]}f_\psi(x)\mathrm{d}x+\sup_{\gamma\in\Gamma_{\bar f}}\int_{\Omega[\bar f=c_{\bar f}]}\gamma(x)f_\psi(x)\mathrm{d}x,
\end{aligned}$$

where in above the last step, we applied Lemma 3.33 and used the fact

$$\Gamma_{\bar f}=\left\{\gamma\in L^\infty(\Omega[\bar f=c_{\bar f}];[0,1])\,\middle|\,\int_{\Omega[\bar f=c_{\bar f}]}\gamma(x)\mathrm{d}x=(\alpha-\underline{\alpha}_{\bar f})\cdot m(\Omega)\right\}.$$

This, together with (3.73), implies that

$$\int_{\Omega[\bar f>c_{\bar f}]}f_\psi(x)\mathrm{d}x+\sup_{\gamma\in\Gamma_{\bar f}}\int_{\Omega[\bar f=c_{\bar f}]}\gamma(x)f_\psi(x)\mathrm{d}x+\langle\, y_0,\psi(0)\,\rangle=0,\ \forall\,\psi\in\overline{Y_2}. \tag{3.75}$$

For any $\hat\psi\in\overline{Y_2}$, it follows from (3.75) that

$$\begin{aligned}
&\int_{\Omega[\bar f>c_{\bar f}]}\mathrm{d}x\int_0^T\bar\psi(x,t)\hat\psi(x,t)\mathrm{d}t+\sup_{\gamma\in\Gamma_{\bar f}}\int_{\Omega[\bar f=c_{\bar f}]}\gamma(x)\mathrm{d}x\int_0^T\bar\psi(x,t)\hat\psi(x,t)\mathrm{d}t\\
&+\langle\, y_0,\bar\psi(0)\,\rangle=0;\\
&-\int_{\Omega[\bar f>c_{\bar f}]}\mathrm{d}x\int_0^T\bar\psi(x,t)\hat\psi(x,t)\mathrm{d}t+\sup_{\gamma\in\Gamma_{\bar f}}\int_{\Omega[\bar f=c_{\bar f}]}\gamma(x)\mathrm{d}x\int_0^T-\bar\psi(x,t)\hat\psi(x,t)\mathrm{d}t\\
&-\langle\, y_0,\bar\psi(0)\,\rangle=0.
\end{aligned}\tag{3.76}$$

Therefore,

$$\sup_{\gamma\in\Gamma_{\bar f}}\int_{\Omega[\bar f=c_{\bar f}]}\gamma(x)\mathrm{d}x\int_0^T\bar\psi(x,t)\hat\psi(x,t)\mathrm{d}t=\inf_{\gamma\in\Gamma_{\bar f}}\int_{\Omega[\bar f=c_{\bar f}]}\gamma(x)\mathrm{d}x\int_0^T\bar\psi(x,t)\hat\psi(x,t)\mathrm{d}t,$$

and

$$\int_0^T\bar\psi(x,t)\hat\psi(x,t)\mathrm{d}t\text{ is identical to be constant in }\Omega[\bar f=c_{\bar f}]\text{ for any fixed }\hat\psi\in\overline{Y_2}. \tag{3.77}$$

Step 2. We claim that

$$(\hat\theta,\bar\psi)\text{ is a Nash equilibrium to problem (GP4) (3.69) for any }\hat\theta\in\mathcal{O}(\bar f). \tag{3.78}$$

To obtain (3.78), it follows from (3.43) that we need only to prove that $\bar{\psi}$ solves the following problem:

$$\inf_{\psi\in\overline{Y_{\beta,2}}}\left[\frac{1}{2}\int_0^T\int_\Omega\hat{\theta}(x)\psi(x,t)^2\mathrm{d}x\mathrm{d}t+\langle\, y_0,\psi(0)\,\rangle\right], \tag{3.79}$$

or equivalently, $\beta\bar{\psi}$ solves

$$\inf_{\zeta\in\beta\overline{Y_{\beta,2}}}\left[\frac{1}{2}\|\zeta\|^2_{L^2(0,T;L^2(\Omega))}+\langle\mathcal{T}_{\beta,2}\,y_0,\,\zeta\rangle\right], \tag{3.80}$$

where $\beta=\hat{\theta}^{1/2}\in\mathcal{B}$.

On the other hand, since (3.80) is a quadratic optimization problem, $\bar{\psi}$ is a solution if and only if $\bar{\psi}$ satisfies the following Euler-Lagrange equation:

$$\int_\Omega\hat{\theta}(x)\mathrm{d}x\int_0^T\bar{\psi}(x,t)\psi(x,t)\mathrm{d}t+\langle\, y_0,\psi(0)\,\rangle=0,\forall\,\psi\in\overline{Y_{\beta,2}}.$$

Since $Y\subset\overline{Y_2}\subset\overline{Y_{\beta,2}}$, and Y is dense in $\overline{Y_{\beta,2}}$, we have

$$\int_\Omega\hat{\theta}(x)\mathrm{d}x\int_0^T\bar{\psi}(x,t)\psi(x,t)\mathrm{d}t+\langle\, y_0,\psi(0)\,\rangle=0,\ \forall\,\psi\in\overline{Y_2}. \tag{3.81}$$

To show that $\bar{\psi}$ is a solution, we only need to prove (3.81). By (3.67),

$$\hat{\theta}(x)=1\text{ when }x\in\Omega[\bar{f}>c_{\bar{f}}],\ \hat{\theta}(x)=0\text{ when }x\in\Omega[\bar{f}<c_{\bar{f}}].$$

Thus (3.81) can be written as

$$\int_{\Omega[\bar{f}>c_{\bar{f}}]}f_\psi(x)\mathrm{d}x+\int_{\Omega[\bar{f}=c_{\bar{f}}]}\hat{\theta}(x)f_\psi(x)\mathrm{d}x+\langle\, y_0,\psi(0)\,\rangle=0,\ \forall\,\psi\in\overline{Y_2}.$$

By $\hat{\theta}\in\mathcal{O}(\bar{f})$ and (3.65), it follows that

$$\chi_{\Omega[\bar{f}=c_{\bar{f}}]}\hat{\theta}\in\Gamma_{\bar{f}}.$$

This, together with (3.77), implies that

$$\int_{\Omega[\bar{f}=c_{\bar{f}}]}\hat{\theta}(x)f_\psi(x)\mathrm{d}x=\sup_{\gamma\in\Gamma_{\bar{f}}}\int_{\Omega[\bar{f}=c_{\bar{f}}]}\gamma(x)f_\psi(x)\mathrm{d}x,\ \forall\,\psi\in\overline{Y_2}. \tag{3.82}$$

Equation (3.81) then follows from (3.75) and (3.82). That is, $\bar{\psi}$ is a solution to (3.79). This proves (3.78).

Finally, it follows from Theorem 3.28 that $\hat{\beta}=\hat{\theta}^{1/2}$ is a relaxed optimal actuator location. □

Denote

$$\operatorname{supp}\theta=\left\{x\in\Omega\;\middle|\;\theta(x)\neq 0\right\},\ \forall\,\theta\in L^1(\Omega). \tag{3.83}$$

Lemma 3.33. *Let $\Omega[\phi\geq c]$, $\Omega[\phi=c]$, $\Omega[\phi<c]$, and let $M_\phi(c)$, c_ϕ, $\bar{\alpha}_\phi$, $\underline{\alpha}_\phi$, $\bar{f}$, f_ψ be defined by (3.52), (3.53), (3.55), (3.58), (3.74), respectively. Then*

$$\begin{aligned}&\lim_{\varepsilon\to0+}\frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta}\int_\Omega\theta(x)\left[\bar{f}(x)+2\varepsilon f_\psi(x)\right]\mathrm{d}x-\max_{\theta\in\Theta}\int_\Omega\theta(x)\bar{f}(x)\mathrm{d}x\right)\\&=\int_{\Omega[\bar{f}>c_{\bar{f}}]}f_\psi(x)\mathrm{d}x+\sup_{\gamma\in\Gamma_{\bar{f}}}\int_{\Omega[\bar{f}=c_{\bar{f}}]}\gamma(x)f_\psi(x)\mathrm{d}x,\end{aligned}$$

where

$$\Gamma_{\bar{f}} = \left\{ \gamma \in L^\infty(\Omega[\bar{f} = c_{\bar{f}}]; [0,1]) \,\middle|\, \int_{\Omega[\bar{f}=c_{\bar{f}}]} \gamma(x)\mathrm{d}x = (\alpha - \underline{\alpha}_{\bar{f}}) \cdot m(\Omega) \right\}. \tag{3.84}$$

To prove this lemma, we need the following results.

Lemma 3.34. *Let $\Omega[\phi \geq c]$, $\Omega[\phi = c]$, $\Omega[\phi < c]$, and let $M_\phi(c)$, c_ϕ, $\bar{\alpha}_\phi$, $\underline{\alpha}_\phi$, $\bar{f}$, f_ψ $\Gamma_{\bar{f}}$ be defined by (3.52), (3.53), (3.55), (3.58), (3.74), (3.84) respectively. Then*

$$\begin{aligned} &\varliminf_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta} \int_\Omega \theta(x) \left[\bar{f}(x) + 2\varepsilon f_\psi(x)\right] \mathrm{d}x - \max_{\theta\in\Theta}\int_\Omega \theta(x)\bar{f}(x)\mathrm{d}x \right) \\ \geq\ & \int_{\Omega[\bar{f}>c_{\bar{f}}]} f_\psi(x)\mathrm{d}x + \sup_{\gamma\in\Gamma_{\bar{f}}} \int_{\Omega[\bar{f}=c_{\bar{f}}]} \gamma(x) f_\psi(x)\mathrm{d}x. \end{aligned} \tag{3.85}$$

Proof. Considering $\Omega[\bar{f} = c_{\bar{f}}]$ in (3.85) as Ω in problem (3.51), and noticing $m(\Omega[\bar{f} = c_{\bar{f}}]) \geq (\alpha - \underline{\alpha}_{\bar{f}})m(\Omega)$, we obtain, from Lemma 3.30, that

$$\sup_{\gamma\in\Gamma_{\bar{f}}} \int_{\Omega[\bar{f}=c_{\bar{f}}]} \gamma(x) f_\psi(x)\mathrm{d}x$$

admits a solution which is denoted as $\bar{\gamma}$, i.e.

$$\int_{\Omega[\bar{f}=c_{\bar{f}}]} \bar{\gamma}(x) f_\psi(x)\mathrm{d}x = \max_{\gamma\in\Gamma_{\bar{f}}} \int_{\Omega[\bar{f}=c_{\bar{f}}]} \gamma(x) f_\psi(x)\mathrm{d}x. \tag{3.86}$$

We claim that $\bar{\theta} = \chi_{\Omega[\bar{f}>c_{\bar{f}}]} + \bar{\gamma}\cdot\chi_{\Omega[\bar{f}=c_{\bar{f}}]} \in \Theta$, and

$$\int_\Omega \bar{\theta}(x)\bar{f}(x)\mathrm{d}x = \max_{\theta\in\Theta}\int_\Omega \theta(x)\bar{f}(x)\mathrm{d}x. \tag{3.87}$$

Actually, by (3.60), it follows that

$$m(\Omega[\bar{f} > c_{\bar{f}}]) = \underline{\alpha}_{\bar{f}} \cdot m(\Omega).$$

This, together with $\bar{\gamma} \in \Gamma_{\bar{f}}$, implies that $\bar{\theta} \in \Theta$. So the claim follows from Lemma 3.30.

By virtue of (3.87) and (3.86), we have

$$\begin{aligned} &\varliminf_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta} \int_\Omega \theta(x) \left(\bar{f}(x) + 2\varepsilon f_\psi(x)\right) \mathrm{d}x - \max_{\theta\in\Theta}\int_\Omega \theta(x)\bar{f}(x)\mathrm{d}x \right) \\ &\geq \lim_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\left(\int_\Omega \bar{\theta}(x) \left(\bar{f}(x) + 2\varepsilon f_\psi(x)\right) \mathrm{d}x - \int_\Omega \bar{\theta}(x)\bar{f}(x)\mathrm{d}x \right) \\ &= \int_\Omega \bar{\theta}(x) f_\psi(x)\mathrm{d}x = \int_{\Omega[\bar{f}>c_{\bar{f}}]} f_\psi(x)\mathrm{d}x + \max_{\gamma\in\Gamma_{\bar{f}}} \int_{\Omega[\bar{f}=c_{\bar{f}}]} \gamma(x) f_\psi(x)\mathrm{d}x. \end{aligned}$$

This proves inequality (3.85). □

To estimate

$$\varlimsup_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta} \int_\Omega \theta(x) \left[\bar{f}(x) + 2\varepsilon f_\psi(x)\right] \mathrm{d}x - \max_{\theta\in\Theta}\int_\Omega \theta(x)\bar{f}(x)\mathrm{d}x \right),$$

we need the following Lemma 3.35.

Lemma 3.35. *Let* $\Omega[\phi \geq c]$, $\Omega[\phi = c]$, $\Omega[\phi < c]$, *and let* $M_\phi(c)$, c_ϕ, $\bar{\alpha}_\phi$, $\underline{\alpha}_\phi$, $\bar{f}$, f_ψ *be defined by (3.52), (3.53), (3.55), (3.58), (3.74), respectively. Let* $\delta > 0$ *and denote by*

$$I^\delta = \overline{\lim_{\varepsilon\to 0+}} \int_{\Omega^{\delta,\varepsilon}} \theta^\varepsilon(x) f_\psi(x) \mathrm{d}x, \quad II^\delta = \overline{\lim_{\varepsilon\to 0+}} \int_{\hat{\Omega}^{\delta,\varepsilon}\setminus\Omega^{\delta,\varepsilon}} \theta^\varepsilon(x) f_\psi(x) \mathrm{d}x. \tag{3.88}$$

Then

$$\overline{\lim_{\varepsilon\to 0+}} \frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta} \int_\Omega \theta(x) \left[\bar{f}(x) + 2\varepsilon f_\psi(x)\right] \mathrm{d}x - \max_{\theta\in\Theta} \int_\Omega \theta(x)\bar{f}(x)\mathrm{d}x \right) \leq I^\delta + II^\delta, \tag{3.89}$$

In addition,

$$I^\delta \leq \int_{\Omega[\bar{f}\geq c_{\bar{f}}+6\delta]} f_\psi(x)\mathrm{d}x. \tag{3.90}$$

Proof. Let $\delta > 0$ be fixed and denote $f^\varepsilon = \bar{f} + 2\varepsilon f_\psi$ for any $\varepsilon > 0$. Notice that

$$\begin{aligned} \Omega^{\delta,\varepsilon} \triangleq\ & \Omega[\bar{f} \geq c_{\bar{f}} + 6\delta] \bigcap \Omega[f_\psi \geq -\delta/\varepsilon] \\ \subset\ & \Omega[f^\varepsilon \geq c_{\bar{f}} + 4\delta] \\ \subset\ & \Omega[\bar{f}(x) \geq c_{\bar{f}} + 2\delta] \bigcup \Omega[f_\psi \geq \delta/\varepsilon]. \end{aligned} \tag{3.91}$$

It follows from (3.91) and (3.57) that

$$\begin{aligned} & \overline{\lim_{\varepsilon\to 0+}}\, m(\Omega[f^\varepsilon \geq c_{\bar{f}} + 4\delta]) \\ \leq\ & \overline{\lim_{\varepsilon\to 0+}} \left(m(\Omega[\bar{f} \geq c_{\bar{f}} + 2\delta]) + m(\Omega[f_\psi \geq \delta/\varepsilon])\right) \\ \leq\ & m\left(\Omega[\bar{f} \geq c_{\bar{f}} + 2\delta]\right) < \alpha m(\Omega). \end{aligned}$$

So there is $\varepsilon(\delta) > 0$ such that

$$m(\Omega[f^\varepsilon \geq c_{\bar{f}} + 4\delta]) < \alpha m(\Omega), \ \forall\, \varepsilon \leq \varepsilon(\delta). \tag{3.92}$$

We claim that : for $\varepsilon \in (0, \varepsilon(\delta))$, if $\theta^\varepsilon \in \mathcal{O}(f^\varepsilon)$, i.e. $\theta^\varepsilon \in \Theta$ solves

$$\max_{\theta\in\Theta} \int_\Omega \theta(x) \left(\bar{f}(x) + 2\varepsilon f_\psi(x)\right) \mathrm{d}x,$$

then

$$\theta^\varepsilon \geq \chi_{\Omega^{\delta,\varepsilon}}. \tag{3.93}$$

Actually, by (3.91), we only need to prove that

$$\theta^\varepsilon \geq \chi_{E_0} \text{ with } E_0 \triangleq \Omega[f^\varepsilon \geq c_{\bar{f}} + 4\delta].$$

If this is false, then there exist $\theta_0 \in \mathcal{O}(f^\varepsilon)$ and $E_1 \subset E_0$ such that

$$m(E_1) > 0 \text{ and } \theta_0(x) < 1, \quad \text{a.e. } x \in E_1. \tag{3.94}$$

By (3.94) and (3.92),

$$\lambda = \min\left\{ \int_{E_1} (1 - \theta_0(x))\mathrm{d}x, \int_{\Omega\setminus E_0} \theta_0(x)\mathrm{d}x \right\} > 0.$$

So there are $E_2 \subset E_1$ and $E_3 \subset \Omega \setminus E_0$ such that

$$\int_{E_2} (1 - \theta_0(x))\mathrm{d}x = \int_{E_3} \theta_0(x)\mathrm{d}x = \lambda. \tag{3.95}$$

Take

$$\tilde{\theta}_0 = \chi_{\Omega\setminus(E_2\cup E_3)}\theta_0 + \chi_{E_2}.$$

It follows from (3.95) that

$$\begin{aligned}\int_\Omega \tilde{\theta}_0(x)\mathrm{d}x &= \int_\Omega \left(\chi_{\Omega\setminus(E_2\cup E_3)}\theta_0(x) + \chi_{E_2}\right)\mathrm{d}x \\ &= \int_\Omega \theta_0(x)\mathrm{d}x - \int_{E_2} \theta_0(x)\mathrm{d}x - \int_{E_3} \theta_0(x)\mathrm{d}x + \int_{E_2} \mathrm{d}x \\ &= \int_\Omega \theta_0(x)\mathrm{d}x = \alpha m(\Omega),\end{aligned} \tag{3.96}$$

i.e., $\tilde{\theta}_0 \in \Theta$. Moreover, by recalling $E_0 = \Omega[f^\varepsilon \geq c_{\bar{f}} + 4\delta]$, we have

$$f^\varepsilon(x) \geq c_{\bar{f}} + 4\delta, \quad \text{a.e. } x \in E_2 \subset E_0.$$

This, together with (3.95), yields

$$\begin{aligned}&\int_\Omega \tilde{\theta}_0(x)f^\varepsilon(x)\mathrm{d}x = \int_\Omega \left(\chi_{\Omega\setminus(E_2\cup E_3)}\theta_0 + \chi_{E_2}\right)(x)f^\varepsilon(x)\mathrm{d}x \\ &= \int_{\Omega\setminus(E_2\cup E_3)} \theta_0(x)f^\varepsilon(x)\mathrm{d}x + \int_{E_2} f^\varepsilon(x)\mathrm{d}x = \int_{\Omega\setminus E_3} \theta_0(x)f^\varepsilon(x)\mathrm{d}x + \int_{E_2} (1 - \theta_0(x))f^\varepsilon(x)\mathrm{d}x \\ &\geq \int_{\Omega\setminus E_3} \theta_0(x)f^\varepsilon(x)\mathrm{d}x + \int_{E_2} (1 - \theta_0(x))\left(c_{\bar{f}} + 4\delta\right)\mathrm{d}x \\ &= \int_{\Omega\setminus E_3} \theta_0 f^\varepsilon(x)\mathrm{d}x + \left(c_{\bar{f}} + 4\delta\right)\int_{E_3} \theta_0(x)\mathrm{d}x.\end{aligned}$$

Note that

$$f^\varepsilon(x) < c_{\bar{f}} + 4\delta, \quad \text{a.e. } x \in E_3 \subset \Omega \setminus E_0.$$

The above two inequalities imply that

$$\int_\Omega \tilde{\theta}_0(x)f^\varepsilon(x)\mathrm{d}x > \int_{\Omega\setminus E_3} \theta_0(x)f^\varepsilon(x)\mathrm{d}x + \int_{E_3} \theta_0(x)f^\varepsilon(x)\mathrm{d}x = \int_\Omega \theta_0(x)f^\varepsilon(x)\mathrm{d}x.$$

By $\tilde{\theta}_0 \in \Theta$ from (3.96), the above inequality contradicts with $\theta \in \mathcal{O}(f^\varepsilon)$. Thus the claim follows.

Set

$$d_{\bar{f}} = \inf\left\{d \in \mathbb{R} \mid m(\Omega[\bar{f} \leq d]) \geq (1 - \alpha)m(\Omega)\right\}. \tag{3.97}$$

Since for any $r > 0$, $m(\Omega[\bar{f} \leq d_{\bar{f}} - r]) < (1 - \alpha)m(\Omega)$, so

$$m(\Omega[\bar{f} \geq d_{\bar{f}} - r]) \geq m(\Omega[\bar{f} > d_{\bar{f}} - r]) > \alpha m(\Omega). \tag{3.98}$$

This, together with (3.55), implies that $d_{\bar{f}} - r \leq c_{\bar{f}}$ for all $r > 0$, and hence $c_{\bar{f}} \geq d_{\bar{f}}$.

Since

$$\Omega[\bar f \le d] = \bigcap_{\varepsilon>0} \Omega[\bar f \le d+\varepsilon],$$

it has

$$m(\Omega[\bar f \le d]) = \lim_{\varepsilon\to 0+} m(\Omega[\bar f \le d+\varepsilon]).$$

By (3.97), the infimum defining $d_{\bar f}$ can be reached. Thus

$$m(\Omega[\bar f \le d_{\bar f}]) \ge (1-\alpha)m(\Omega).$$

By the definition of $c_{\bar f}$, $m(\Omega[\bar f \ge c_{\bar f}]) \ge \alpha m(\Omega)$. Therefore,

$$\begin{aligned} & m(\{x\in\Omega \mid d_{\bar f} < \bar f(x) < c_{\bar f}\}) = m(\Omega) - m(\Omega[\bar f \le d_{\bar f}]) - m(\Omega[\bar f \ge c_{\bar f}]) \\ & \le m(\Omega) - (1-\alpha)m(\Omega) - \alpha m(\Omega) = 0. \end{aligned}$$

i.e.,

$$m\left(\left\{x\in\Omega \mid d_{\bar f} < \bar f(x) < c_{\bar f}\right\}\right) = 0. \tag{3.99}$$

Furthermore,

$$\begin{aligned} & \Omega[\bar f \ge d_{\bar f} - 2\delta] \bigcap \Omega[f_\psi \ge -\delta/\varepsilon] \\ & \subset \Omega[f^\varepsilon \ge d_{\bar f} - 4\delta] \\ & \subset \Omega[\bar f(x) \ge d_{\bar f} - 6\delta] \bigcup \Omega[f_\psi(x) \ge \delta/\varepsilon]. \end{aligned} \tag{3.100}$$

It then follows from (3.100) and (3.98) that

$$\begin{aligned} & \varliminf_{\varepsilon\to 0+} m\left(\Omega[f^\varepsilon \ge d_{\bar f} - 4\delta]\right) \\ \ge \; & \varliminf_{\varepsilon\to 0+} m\left(\Omega[\bar f \ge d_{\bar f} - 2\delta] \bigcap \Omega[f_\psi \ge -\delta/\varepsilon]\right) \\ = \; & m\left(\Omega[\bar f \ge d_{\bar f} - 2\delta]\right) > \alpha m(\Omega). \end{aligned}$$

So there is $\hat\varepsilon(\delta) > 0$ such that

$$m\left(\Omega[f^\varepsilon \ge d_{\bar f} - 4\delta]\right) > \alpha m(\Omega),\ \forall\, \varepsilon \le \hat\varepsilon(\delta). \tag{3.101}$$

Let

$$\hat\Omega^{\delta,\varepsilon} \triangleq \Omega[\bar f(x) \ge d_{\bar f} - 6\delta] \bigcup \Omega[f_\psi(x) \ge \delta/\varepsilon].$$

Similarly to the proof of Claim 2, we have from (3.101) and (3.100) that for any $\theta^\varepsilon \in \mathcal{O}(f^\varepsilon)$,

$$\theta^\varepsilon \le \chi_{\hat\Omega^{\delta,\varepsilon}} \text{ when } \varepsilon \in (0, \hat\varepsilon(\delta)). \tag{3.102}$$

Choosing ε to satisfy

$$0 < \varepsilon < \min\{\varepsilon(\delta), \hat\varepsilon(\delta)\},$$

it follows from (3.102) and (3.88) that

$$
\begin{aligned}
& \overline{\lim_{\varepsilon\to 0+}} \frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta}\int_\Omega \theta(x)\left(\bar{f}(x)+2\varepsilon f_\psi(x)\right)\mathrm{d}x - \max_{\theta\in\Theta}\int_\Omega \theta(x)\bar{f}(x)\mathrm{d}x\right) \\
\leq \; & \overline{\lim_{\varepsilon\to 0+}} \frac{1}{2\varepsilon}\left(\int_\Omega \theta^\varepsilon(x)\left(\bar{f}(x)+2\varepsilon f_\psi(x)\right)\mathrm{d}x - \int_\Omega \theta^\varepsilon(x)\bar{f}(x)\mathrm{d}x\right) \\
= \; & \overline{\lim_{\varepsilon\to 0+}} \int_\Omega \theta^\varepsilon(x) f_\psi(x)\mathrm{d}x = \overline{\lim_{\varepsilon\to 0+}} \int_{\hat{\Omega}^{\delta,\varepsilon}} \theta^\varepsilon(x) f_\psi(x)\mathrm{d}x \\
\leq \; & \overline{\lim_{\varepsilon\to 0+}} \int_{\Omega^{\delta,\varepsilon}} \theta^\varepsilon(x) f_\psi(x)\mathrm{d}x + \overline{\lim_{\varepsilon\to 0+}} \int_{\hat{\Omega}^{\delta,\varepsilon}\setminus\Omega^{\delta,\varepsilon}} \theta^\varepsilon(x) f_\psi(x)\mathrm{d}x = I^\delta + II^\delta.
\end{aligned}
$$

Thus the inequality (3.89) holds. Now, by (3.93), $\theta^\varepsilon(x) = 1$ for almost all $x \in \Omega^{\delta,\varepsilon}$. It then follows from (3.91) and the dominated convergent theorem that

$$
I^\delta = \overline{\lim_{\varepsilon\to 0+}} \int_{\Omega[\bar{f}\geq c_{\bar{f}}+6\delta]} \chi_{[f_\psi\geq -\delta/\varepsilon]} f_\psi(x)\mathrm{d}x \leq \int_{\Omega[\bar{f}\geq c_{\bar{f}}+6\delta]} f_\psi(x)\mathrm{d}x.
$$

Thus the inequality (3.90) holds and the proof is over. $\square$

Lemma 3.36. *Let $\Omega[\phi \geq c]$, $\Omega[\phi = c]$, $\Omega[\phi < c]$, and let $M_\phi(c)$, c_ϕ, $\bar{\alpha}_\phi$, $\underline{\alpha}_\phi$, $\bar{f}$, f_ψ $\Gamma_{\bar{f}}$ be defined by (3.52), (3.53), (3.55), (3.58), (3.74), (3.84) respectively. Then*

$$
\begin{aligned}
& \overline{\lim_{\varepsilon\to 0+}} \frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta}\int_\Omega \theta(x)\left[\bar{f}(x)+2\varepsilon f_\psi(x)\right]\mathrm{d}x - \max_{\theta\in\Theta}\int_\Omega \theta(x)\bar{f}(x)\mathrm{d}x\right) \\
\leq \; & \int_{\Omega[\bar{f}>c_{\bar{f}}]} f_\psi(x)\mathrm{d}x + \max_{\gamma\in\Gamma_{\bar{f}}} \int_{\Omega[\bar{f}=c_{\bar{f}}]} \gamma(x) f_\psi(x)\mathrm{d}x.
\end{aligned}
\tag{3.103}
$$

Proof. By the definition of $c_{\bar{f}}$ given in (3.55),

$$
\text{either } m(\Omega[\bar{f} \geq c_{\bar{f}}]) > \alpha m(\Omega) \text{ or } m(\Omega[\bar{f} \geq c_{\bar{f}}]) = \alpha m(\Omega).
$$

This, together with $c_{\bar{f}} \geq d_{\bar{f}}$, implies that there are three possible cases:

a) $c_{\bar{f}} = d_{\bar{f}}$;

b) $c_{\bar{f}} > d_{\bar{f}}$ and $m(\Omega[\bar{f} \geq c_{\bar{f}}]) = \alpha m(\Omega)$;

c) $c_{\bar{f}} > d_{\bar{f}}$ and $m(\Omega[\bar{f} \geq c_{\bar{f}}]) > \alpha m(\Omega)$.

First, we exclude the case c). We suppose that this case is true and obtain a contradiction. Actually, by definition (3.97) for $d_{\bar{f}}$,

$$
m(\Omega[\bar{f} \leq (c_{\bar{f}} + d_{\bar{f}})/2]) \geq (1-\alpha)m(\Omega).
$$

This, together with $m(\Omega[\bar{f} \geq c_{\bar{f}}]) > \alpha m(\Omega)$, implies that

$$
m(\Omega) \geq m\big(\Omega[\bar{f}(x) \leq (c_{\bar{f}} + d_{\bar{f}})/2]\big) + m\big(\Omega[\bar{f}(x) \geq c_{\bar{f}}]\big) > (1-\alpha)m(\Omega) + \alpha m(\Omega) = m(\Omega).
$$

So the case c) is impossible. We only discuss the first two cases.

Second, we discuss the case a). Notice that

$$\lim_{\varepsilon\to 0+}\hat{\Omega}^{\delta,\varepsilon}=\Omega[\bar{f}\geq c_{\bar{f}}-6\delta],\quad \lim_{\varepsilon\to 0+}\Omega^{\delta,\varepsilon}=\Omega[\bar{f}\geq c_{\bar{f}}+6\delta].$$

Setting

$$\tilde{\Omega}^{\delta}\triangleq\Omega\big[c_{\bar{f}}-6\delta\leq\bar{f}<c_{\bar{f}}+6\delta\big], \tag{3.104}$$

we have

$$\lim_{\varepsilon\to 0+}\hat{\Omega}^{\delta,\varepsilon}\setminus\Omega^{\delta,\varepsilon}=\tilde{\Omega}^{\delta}.$$

So

$$\chi_{\hat{\Omega}^{\delta,\varepsilon}\setminus\Omega^{\delta,\varepsilon}}\to\chi_{\tilde{\Omega}^{\delta}}\text{ strongly in }L^1(\Omega), \tag{3.105}$$

and

$$\chi_{\hat{\Omega}^{\delta,\varepsilon}\setminus\Omega^{\delta,\varepsilon}}f_{\psi}\to\chi_{\tilde{\Omega}^{\delta}}f_{\psi}\text{ strongly in }L^1(\Omega). \tag{3.106}$$

Suppose that there is a sequence $\{\varepsilon_n^{\delta},\,n\in\mathbb{N}\}$ converging to zero such that

$$\lim_{n\to\infty}\int_{\hat{\Omega}^{\delta,\varepsilon_n^{\delta}}\setminus\Omega^{\delta,\varepsilon_n^{\delta}}}\theta^{\varepsilon_n^{\delta}}(x)f_{\psi}(x)\mathrm{d}x=\overline{\lim_{\varepsilon\to 0+}}\int_{\hat{\Omega}^{\delta,\varepsilon}\setminus\Omega^{\delta,\varepsilon}}\theta^{\varepsilon}(x)f_{\psi}(x)\mathrm{d}x.$$

Since $\{\theta^{\varepsilon_n^{\delta}}\}\subset L^{\infty}(\Omega;[0,1])$, there is a subsequence, still denoted by itself without confusion, such that

$$\theta^{\varepsilon_n^{\delta}}\to\tilde{\theta}^{\delta}\text{ weakly* in }L^{\infty}(\Omega), \tag{3.107}$$

and $\tilde{\theta}^{\delta}\in L^{\infty}(\Omega;[0,1])$. This, together with (3.106), implies that

$$\lim_{n\to\infty}\int_{\hat{\Omega}^{\delta,\varepsilon_n^{\delta}}\setminus\Omega^{\delta,\varepsilon_n^{\delta}}}\theta^{\varepsilon_n^{\delta}}(x)f_{\psi}(x)\mathrm{d}x=\lim_{n\to\infty}\int_{\Omega}\theta^{\varepsilon_n^{\delta}}(x)\left(\chi_{\hat{\Omega}^{\delta,\varepsilon_n^{\delta}}\setminus\Omega^{\delta,\varepsilon_n^{\delta}}}f_{\psi}\right)(x)\mathrm{d}x=\int_{\tilde{\Omega}^{\delta}}\tilde{\theta}^{\delta}(x)f_{\psi}(x)\mathrm{d}x.$$

Therefore,

$$II^{\delta}=\overline{\lim_{\varepsilon\to 0+}}\int_{\hat{\Omega}^{\delta,\varepsilon}\setminus\Omega^{\delta,\varepsilon}}\theta^{\varepsilon}(x)f_{\psi}(x)\mathrm{d}x=\int_{\tilde{\Omega}^{\delta}}\tilde{\theta}^{\delta}(x)f_{\psi}(x)\mathrm{d}x. \tag{3.108}$$

On the other hand, by (3.102), $\operatorname{supp}\theta^{\varepsilon}\subset\hat{\Omega}^{\delta,\varepsilon}$, and so

$$\alpha m(\Omega)=\int_{\Omega}\theta^{\varepsilon}(x)\mathrm{d}x=\int_{\operatorname{supp}\theta^{\varepsilon}}\theta^{\varepsilon}(x)\mathrm{d}x=\int_{\hat{\Omega}^{\delta,\varepsilon}}\theta^{\varepsilon}(x)\mathrm{d}x.$$

This, together with the fact

$$\chi_{\hat{\Omega}^{\delta,\varepsilon_n^{\delta}}}\to\chi_{\Omega[\bar{f}\geq c_{\bar{f}}-6\delta]}\text{ strongly in }L^1(\Omega),$$

implies that

$$\int_{\Omega[\bar{f}\geq c_{\bar{f}}-6\delta]}\tilde{\theta}^{\delta}(x)\mathrm{d}x=\lim_{n\to\infty}\int_{\hat{\Omega}^{\delta,\varepsilon_n^{\delta}}}\theta^{\varepsilon_n^{\delta}}(x)\mathrm{d}x=\alpha m(\Omega). \tag{3.109}$$

Now, we claim that

$$\tilde{\theta}^{\delta}\geq\chi_{\Omega[\bar{f}\geq c_{\bar{f}}+6\delta]}. \tag{3.110}$$

Indeed, since

$$\lim_{n\to\infty}\Omega^{\delta,\varepsilon_n^{\delta}}=\Omega[\bar{f}\geq c_{\bar{f}}+6\delta],$$

it follows from (3.93) that

$$\begin{aligned}\langle g,\, \tilde{\theta}^{\delta}\,\rangle \ &= \lim_{n\to\infty} \langle\, g,\, \theta^{\varepsilon_n^{\delta}}\rangle_{L^1(\Omega),L^\infty(\Omega)} \ge \lim_{n\to\infty}\langle g,\, \chi_{\Omega^{\delta,\varepsilon_n^{\delta}}}\,\rangle \\ &= \langle g,\, \chi_{\Omega[\bar{f}\ge c_{\bar{f}}+6\delta]}\,\rangle, \quad \forall\, g\in L^1(\Omega;[0,\infty)).\end{aligned}$$

Thus (3.110) holds true.

Now recall (3.89). By (3.90) and (3.108),

$$\begin{aligned}&\overline{\lim_{\varepsilon\to 0+}}\, \frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta}\int_\Omega \theta(x)\left(\bar{f}(x)+2\varepsilon f_\psi(x)\right)\mathrm{d}x - \max_{\theta\in\Theta}\int_\Omega \theta(x)\bar{f}(x)\mathrm{d}x\right)\\ \le\ & \int_{\Omega[\bar{f}\ge c_{\bar{f}}+2\delta]} f_\psi(x)\mathrm{d}x + \int_{\tilde{\Omega}^\delta} \tilde{\theta}^\delta(x) f_\psi(x)\mathrm{d}x,\ \forall\,\delta>0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\overline{\lim_{\varepsilon\to 0+}}\, \frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta}\int_\Omega \theta(x)\left(\bar{f}(x)+2\varepsilon f_\psi(x)\right)\mathrm{d}x - \max_{\theta\in\Theta}\int_\Omega \theta(x)\bar{f}(x)\mathrm{d}x\right)\\ \le\ & \underline{\lim_{\delta\to 0+}}\left(\int_{\Omega[\bar{f}\ge c_{\bar{f}}+2\delta]} f_\psi(x)\mathrm{d}x + \int_{\tilde{\Omega}^\delta} \tilde{\theta}^\delta(x) f_\psi(x)\mathrm{d}x\right) \\ =\ & \int_{\Omega[\bar{f}> c_{\bar{f}}]} f_\psi(x)\mathrm{d}x + \underline{\lim_{\delta\to 0+}}\int_{\tilde{\Omega}^\delta} \tilde{\theta}^\delta(x) f_\psi(x)\mathrm{d}x.\end{aligned} \tag{3.111}$$

Suppose that there is a sequence $\{\delta_n\}$ such that

$$\lim_{n\to\infty}\int_{\tilde{\Omega}^{\delta_n}} \tilde{\theta}^{\delta_n}(x) f_\psi(x)\mathrm{d}x = \underline{\lim_{\delta\to 0+}}\int_{\tilde{\Omega}^\delta} \tilde{\theta}^\delta(x) f_\psi(x)\mathrm{d}x.$$

Since $\{\tilde{\theta}^{\delta_n}\}\subset L^\infty(\Omega;[0,1])$, there is a subsequence, still denoted by itself, such that

$$\tilde{\theta}^{\delta_n}\to\bar{\theta}\ \text{weakly* in}\ L^\infty(\Omega),$$

and $\bar{\theta}\in L^\infty(\Omega;[0,1])$. By (3.104),

$$\chi_{\tilde{\Omega}^{\delta_n}} f_\psi \to \chi_{\Omega[\bar{f}=c_{\bar{f}}]} f_\psi\ \text{strongly in}\ L^1(\Omega).$$

Thus

$$\lim_{n\to\infty}\int_{\tilde{\Omega}^{\delta_n}} \tilde{\theta}^{\delta_n}(x) f_\psi(x)\mathrm{d}x = \int_{\Omega[\bar{f}=c_{\bar{f}}]} \bar{\theta}(x) f_\psi(x)\mathrm{d}x. \tag{3.112}$$

With replacement of f_ψ by 1, we can obtain along with (3.109) and (3.110) that

$$\int_{\Omega[\bar{f}\ge c_{\bar{f}}]} \bar{\theta}(x)\mathrm{d}x = \lim_{n\to\infty}\int_{\Omega[\bar{f}\ge c_{\bar{f}}-6\delta_n]} \tilde{\theta}^{\delta_n}(x)\mathrm{d}x = \alpha m(\Omega),$$

and

$$\begin{aligned}m(\Omega[\bar{f}>c_{\bar{f}}]) \ge &\int_{\Omega[\bar{f}>c_{\bar{f}}]} \bar{\theta}(x)\mathrm{d}x = \lim_{n\to\infty}\int_{\Omega[\bar{f}\ge c_{\bar{f}}+6\delta_n]} \tilde{\theta}^{\delta_n}(x)\mathrm{d}x \\ \ge\ & \lim_{n\to\infty}\int_{\Omega[\bar{f}\ge c_{\bar{f}}+6\delta_n]} \mathrm{d}x = m(\Omega[\bar{f}>c_{\bar{f}}]) = \underline{\alpha}_{\bar{f}} m(\Omega).\end{aligned}$$

Therefore,

$$\int_{\Omega[\bar{f}=c_{\bar{f}}]} \bar{\theta}(x)\mathrm{d}x = \int_{\Omega[\bar{f}\geq c_{\bar{f}}]} \bar{\theta}(x)\mathrm{d}x - \int_{\Omega[\bar{f}>c_{\bar{f}}]} \bar{\theta}(x)\mathrm{d}x = (\alpha - \underline{\alpha}_{\bar{f}})m(\Omega).$$

This, together with (3.111) and (3.112), gives inequality (3.103).

Third, we consider the case b) for $c_{\bar{f}} > d_{\bar{f}}$ and $m\left(\Omega[\bar{f}(x) \geq c_{\bar{f}}]\right) = \alpha \cdot m(\Omega)$. By Lemma 3.30, there exists $\theta^{\varepsilon} \in \mathcal{W}$ such that $\theta^{\varepsilon} \in \mathcal{O}(f^{\varepsilon})$. By Lemma 3.30, it follows that

$$\theta^{\varepsilon}(x) \begin{cases} = 1 \text{ when } f^{\varepsilon}(x) > c_{f^{\varepsilon}}; \\ = 0 \text{ or } = 1 \text{ when } f^{\varepsilon}(x) = c_{f^{\varepsilon}}; \\ = 0 \text{ when } f^{\varepsilon}(x) < c_{f^{\varepsilon}}. \end{cases}$$

Hence

$$f^{\varepsilon}(x) \geq c_{f^{\varepsilon}} \geq f^{\varepsilon}(y), \forall\, x \in \operatorname{supp}\theta^{\varepsilon} \text{ a.e. and } y \in \Omega \setminus \operatorname{supp}\theta^{\varepsilon} \text{ a.e.,} \tag{3.113}$$

and

$$m(\operatorname{supp}\theta^{\varepsilon}) = \alpha m(\Omega). \tag{3.114}$$

Let

$$A^{\varepsilon} = \left\{x \in \Omega \;\middle|\; \bar{f}(x) \geq c_{\bar{f}},\, f_{\psi}(x) > -\frac{c_{\bar{f}} - d_{\bar{f}}}{4\varepsilon}\right\}, \quad B^{\varepsilon} = \left\{x \in \Omega \;\middle|\; \bar{f}(x) \leq d_{\bar{f}},\, f_{\psi}(x) < \frac{c_{\bar{f}} - d_{\bar{f}}}{4\varepsilon}\right\}.$$

Then

$$f^{\varepsilon}(x) > \frac{c_{\bar{f}} + d_{\bar{f}}}{2} > f^{\varepsilon}(y), \ \forall\, x \in A^{\varepsilon} \text{ and } y \in B^{\varepsilon} \text{ a.e.} \tag{3.115}$$

Furthermore,

$$\lim_{\varepsilon\to 0} A^{\varepsilon} = \Omega[\bar{f} \geq c_{\bar{f}}], \quad \lim_{\varepsilon\to 0} B^{\varepsilon} = \Omega[\bar{f} \leq d_{\bar{f}}].$$

This, together with (3.99), implies that

$$\lim_{\varepsilon\to 0} m(A^{\varepsilon}) = \alpha \cdot m(\Omega), \ \lim_{\varepsilon\to 0} m(\Omega \setminus (A^{\varepsilon} \cup B^{\varepsilon})) = 0. \tag{3.116}$$

We claim that

$$\lim_{\varepsilon\to 0} m(A^{\varepsilon} \cap \operatorname{supp}\theta^{\varepsilon}) = \alpha \cdot m(\Omega). \tag{3.117}$$

To see this, for each $\varepsilon > 0$, there is at least one of the following two cases to be valid:

$$m\left(A^{\varepsilon} \cap (\Omega \setminus \operatorname{supp}\theta^{\varepsilon})\right) = 0 \text{ or } m\left(B^{\varepsilon} \cap (\operatorname{supp}\theta^{\varepsilon})\right) = 0. \tag{3.118}$$

Otherwise, there exists some $\varepsilon_0 > 0$ such that

$$m\left(A^{\varepsilon_0} \cap (\Omega \setminus \operatorname{supp}\theta^{\varepsilon_0})\right) \neq 0, \ m\left(B^{\varepsilon_0} \cap (\operatorname{supp}\theta^{\varepsilon_0})\right) \neq 0.$$

Notice that

$$A^{\varepsilon_0} \cap (\Omega \setminus \operatorname{supp}\theta^{\varepsilon_0}) \subset \Omega \setminus \operatorname{supp}\theta^{\varepsilon_0}, \ B^{\varepsilon_0} \cap (\operatorname{supp}\theta^{\varepsilon_0}) \subset \operatorname{supp}\theta^{\varepsilon_0}.$$

It follows from (3.113) that

$$f^{\varepsilon_0}(x) \leq f^{\varepsilon_0}(y) \text{ for almost all } x \in A^{\varepsilon_0} \cap (\Omega \setminus \operatorname{supp}\theta^{\varepsilon_0}) \text{ and } y \in B^{\varepsilon_0} \cap (\operatorname{supp}\theta^{\varepsilon_0}).$$

This contradicts with (3.115).

If there is a sequence $\{\varepsilon_n\}$ such that $m\left(A^{\varepsilon_n}\cap(\Omega\setminus\operatorname{supp}\theta^{\varepsilon_n})\right)=0$, then we have (3.117) by virtue of (3.116).

If there is a sequence $\{\varepsilon_n\}$ such that $m\left(B^{\varepsilon_n}\cap(\operatorname{supp}\theta^{\varepsilon_n})\right)=0$, then, for any $n\in\mathbb{N}$,

$$\begin{aligned}
&m\left(B^{\varepsilon_n}\cap(\operatorname{supp}\theta^{\varepsilon_n})\right)=0\Rightarrow m\left(B^{\varepsilon_n}\cap(\Omega\setminus\operatorname{supp}\theta^{\varepsilon_n})\right)=m\left(B^{\varepsilon_n}\right)\\
&\Rightarrow m\left((\Omega\setminus B^{\varepsilon_n})\cup\operatorname{supp}\theta^{\varepsilon_n}\right)=m\left(\Omega\setminus B^{\varepsilon_n}\right)\\
&\Rightarrow m\Big(((\Omega\setminus B^{\varepsilon_n})\cap A^{\varepsilon_n})\cup((\Omega\setminus B^{\varepsilon_n})\setminus A^{\varepsilon_n})\cup\operatorname{supp}\theta^{\varepsilon_n}\Big)=m\Big(((\Omega\setminus B^{\varepsilon_n})\cap A^{\varepsilon_n})\cup((\Omega\setminus B^{\varepsilon_n})\setminus A^{\varepsilon_n})\Big)\\
&\Rightarrow m\left((A^{\varepsilon_n}\cup\operatorname{supp}\theta^{\varepsilon_n})\cup(\Omega\setminus(A^{\varepsilon_n}\cup B^{\varepsilon_n}))\right)=m\left(A^{\varepsilon_n}\cup(\Omega\setminus(A^{\varepsilon_n}\cup B^{\varepsilon_n}))\right),
\end{aligned}$$

where the last assertion above follows from $m\left(A^{\varepsilon_n}\cap B^{\varepsilon_n}\right)=0$ by virtue of (3.115). Furthermore, the last assertion above, likely (3.116), implies that

$$\lim_{n\to\infty}m(A^{\varepsilon_n}\cup\operatorname{supp}\theta^{\varepsilon_n})=\lim_{n\to\infty}m(A^{\varepsilon_n}).$$

Thus

$$\begin{aligned}
\lim_{n\to\infty}m(A^{\varepsilon_n}\cap\operatorname{supp}\theta^{\varepsilon_n})&=\lim_{n\to\infty}m(\operatorname{supp}\theta^{\varepsilon_n})-\lim_{n\to\infty}m(\operatorname{supp}\theta^{\varepsilon_n}\setminus A^{\varepsilon_n})\\
&=\lim_{n\to\infty}m(\operatorname{supp}\theta^{\varepsilon_n})-\Big(\lim_{n\to\infty}m(A^{\varepsilon_n}\cup\operatorname{supp}\theta^{\varepsilon_n})-\lim_{n\to\infty}m(A^{\varepsilon_n})\Big)\\
&=\lim_{n\to\infty}m(\operatorname{supp}\theta^{\varepsilon_n})=\alpha m(\Omega).
\end{aligned}$$

Therefore (3.117) is true.

By (3.116)-(3.117) and (3.114), it holds that

$$\lim_{\varepsilon\to0}m\left(A^{\varepsilon}\setminus\operatorname{supp}\theta^{\varepsilon}\right)=\lim_{\varepsilon\to0}m\left(\operatorname{supp}\theta^{\varepsilon}\setminus A^{\varepsilon}\right)=0.$$

By the absolute continuity of the Lebesgue integral,

$$\lim_{\varepsilon\to0+}\int_{A^{\varepsilon}\setminus\operatorname{supp}\theta^{\varepsilon}}|f_\psi(x)|\mathrm{d}x=\lim_{\varepsilon\to0+}\int_{\operatorname{supp}\theta^{\varepsilon}\setminus A^{\varepsilon}}|f_\psi(x)|\mathrm{d}x=0.$$

Thus

$$\begin{aligned}
&\lim_{\varepsilon\to0+}\left|\int_{\operatorname{supp}\theta^{\varepsilon}}\theta^{\varepsilon}(x)f_\psi(x)\mathrm{d}x-\int_{A^{\varepsilon}}\theta^{\varepsilon}(x)f_\psi(x)\mathrm{d}x\right|\\
\le&\lim_{\varepsilon\to0+}\left[\left|\int_{A^{\varepsilon}\setminus\operatorname{supp}\theta^{\varepsilon}}\theta^{\varepsilon}(x)f_\psi(x)\mathrm{d}x\right|+\left|\int_{\operatorname{supp}\theta^{\varepsilon}\setminus A^{\varepsilon}}\theta^{\varepsilon}(x)f_\psi(x)\mathrm{d}x\right|\right]\\
\le&\lim_{\varepsilon\to0+}\left[\left|\int_{A^{\varepsilon}\setminus\operatorname{supp}\theta^{\varepsilon}}|f_\psi(x)|\mathrm{d}x\right|+\left|\int_{\operatorname{supp}\theta^{\varepsilon}\setminus A^{\varepsilon}}|f_\psi(x)|\mathrm{d}x\right|\right]=0.
\end{aligned}\tag{3.119}$$

With the similar argument, we can prove that

$$\lim_{\varepsilon\to0+}\left|\int_{A^{\varepsilon}}\theta^{\varepsilon}(x)f_\psi(x)\mathrm{d}x-\int_{\Omega[\bar f\ge c_{\bar f}]}\theta^{\varepsilon}(x)f_\psi(x)\mathrm{d}x\right|=0.\tag{3.120}$$

By (3.119) and (3.120),

$$
\begin{aligned}
& \overline{\lim_{\varepsilon\to 0+}} \frac{1}{2\varepsilon}\left(\max_{\theta\in\Theta} \int_\Omega \theta(x)\left(\bar f(x) + 2\varepsilon f_\psi(x)\right)\mathrm{d}x - \max_{\theta\in\Theta}\int_\Omega \theta(x)\bar f(x)\mathrm{d}x\right)\\
\le\ & \overline{\lim_{\varepsilon\to 0+}} \frac{1}{2\varepsilon}\left(\int_\Omega \theta^\varepsilon(x)\left(\bar f(x) + 2\varepsilon f_\psi(x)\right)\mathrm{d}x - \int_\Omega \theta^\varepsilon(x)\bar f(x)\mathrm{d}x\right)\\
=\ & \overline{\lim_{\varepsilon\to 0+}} \int_\Omega \theta^\varepsilon(x) f_\psi(x)\mathrm{d}x = \overline{\lim_{\varepsilon\to 0+}} \int_{\operatorname{supp}\theta^\varepsilon} \theta^\varepsilon(x) f_\psi(x)\mathrm{d}x \qquad\qquad (3.121)\\
=\ & \overline{\lim_{\varepsilon\to 0+}} \int_{A^\varepsilon} \theta^\varepsilon(x) f_\psi(x)\mathrm{d}x = \overline{\lim_{\varepsilon\to 0+}} \int_{\Omega[\bar f\ge c_{\bar f}]} \theta^\varepsilon(x) f_\psi(x)\mathrm{d}x\\
\le\ & \int_{\Omega[\bar f\ge c_{\bar f}]} f_\psi(x)\mathrm{d}x.
\end{aligned}
$$

Moreover, it follows from the assumption $m\left(\Omega[\bar f(x)\ge c_{\bar f}]\right) = \alpha\cdot m(\Omega)$ that $\Gamma_{\bar f}$ is a singleton. Thus (3.121) gives (3.103). □

Proof of Lemma 3.33. Combining Lemmas 3.34 and 3.36, we obtain Lemma 3.33 immediately. □

Proof of Theorem 1.3. By Proposition 3.32 and Lemma 3.30, we see that when $\omega\in\mathcal{W}$, χ_ω solves problem (1.8) if and only if

$$\chi_\omega \in \mathcal{O}(G_{\bar\psi}).$$

Moreover, there must have an $\bar\omega\in\Omega$ such that $\chi_{\bar\omega}$ solves problem (1.8). By the optimality of $\chi_{\bar\omega}$, it follows from (1.8) that

$$N_2(\chi_{\bar\omega}) = \inf_{\beta\in\mathcal{B}} N_2(\beta).$$

Notice that

$$N_2(\chi_{\bar\omega}) \ge \inf_{\omega\in\mathcal{W}} N_2(\chi_\omega),\ \inf_{\omega\in\mathcal{W}} N_2(\chi_\omega) \ge \inf_{\beta\in\mathcal{B}} N_2(\beta).$$

We thus have

$$\inf_{\omega\in\mathcal{W}} N_2(\chi_\omega) = \inf_{\beta\in\mathcal{B}} N_2(\beta).$$

Therefore, if χ_ω solves problem (1.8), then ω must be an optimal actuator location of problem (1.4). Notice that $\chi_\omega\in\mathcal{O}(G_{\bar\psi})$ is equivalent to ω solving

$$\sup_{\omega\in\mathcal{W}} \|\chi_\omega \bar f\|^2_{L^2(\Omega)}, \quad \text{with } \bar f(x) = \int_0^T \bar\psi(x,t)\mathrm{d}t.$$

Using Proposition 3.32 again, we can derive the results of Theorem 1.3. □

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